Lecture two

Objective of Lecture:

- Sampling theorem

1.8 Sampling theorem:

Sampling of the signals is the fundamental operation in digital communication. A continuous time signal is first converted to discrete time signal by sampling process. Also it should be possible to recover or reconstruct the signal completely from its samples.

The sampling theorem state that:

1- *A band limited signal of finite energy, which has no frequency components higher than W Hz, is completely described by specifying the values of the signal at instant of time separated by 1/2W second and*
2- *A band limited signal of finite energy, which has no frequency components higher than W Hz, may be completely recovered from the knowledge of its samples taken at the rate of 2W samples per second.*

Proof of sampling theorem:

Let *x(t)* the continuous time signal shown in figure below, its band width does not contain any frequency components higher than *W* Hz. A sampling function samples this signal regularly at the rate of f_S sample per second.

Assume an analog waveform, $x(t)$ with a Fourier transform, $X(f)$, which is zero outside the interval $(-f_m < f < f_m)$. The sampling of $x(t)$ can viewed as the product of $x(t)$ with periodic train of unit impulse function $x_\delta(t)$ defined as

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

The sifting property of unit impulse state that

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

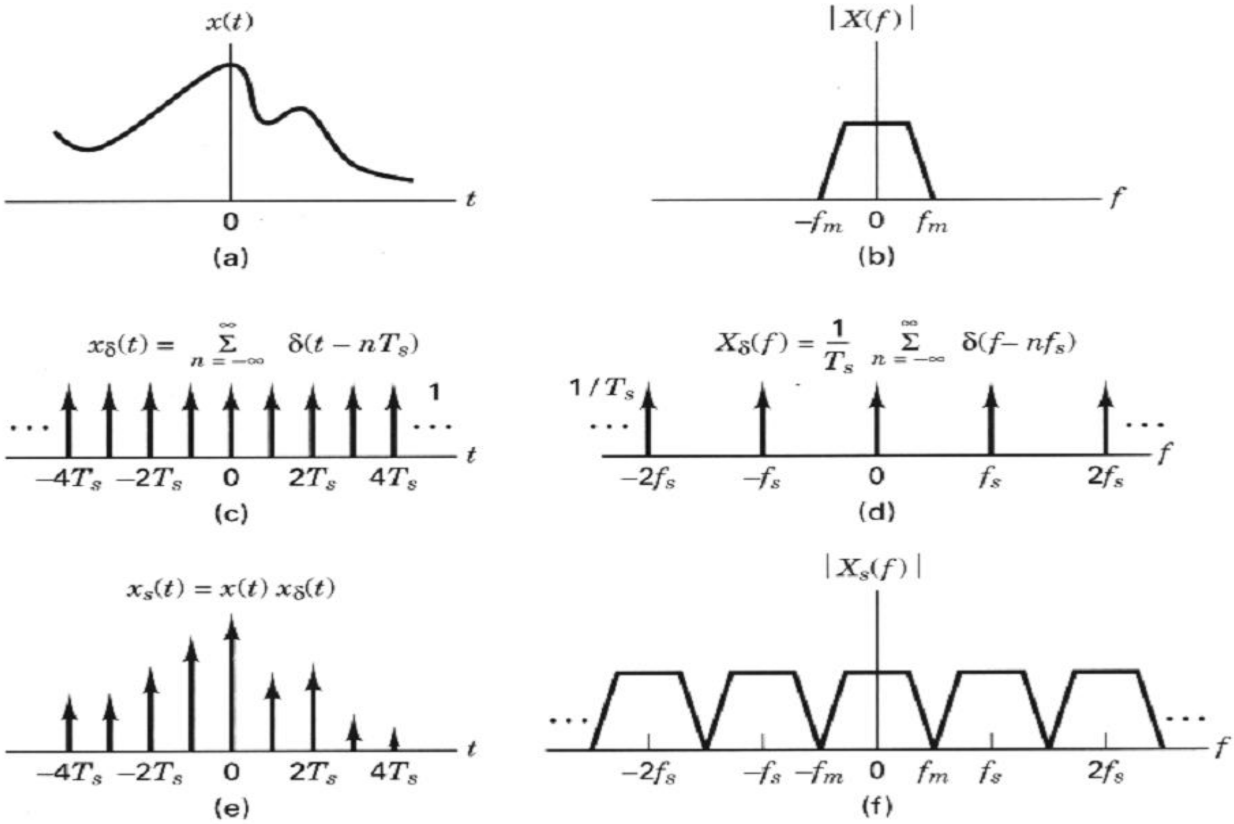
Using this property so that:

$$x_s(t) = x(t)x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Notice that the Fourier transform of an impulse train is another impulse train.

$$X_\delta(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} \delta(f - nf_s)$$



Convolution with an impulse function simply shifts the original function:

$$X(f) * \delta(f - nf_s)$$

We can now solve for the transform $X_s(f)$ of the sampled waveform:

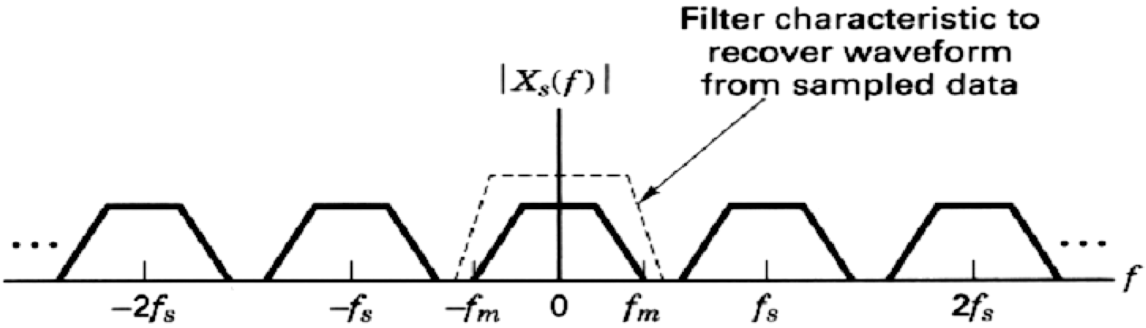
$$X(f) * \delta(f - nf_s) = X(f - nf_s)$$

So that

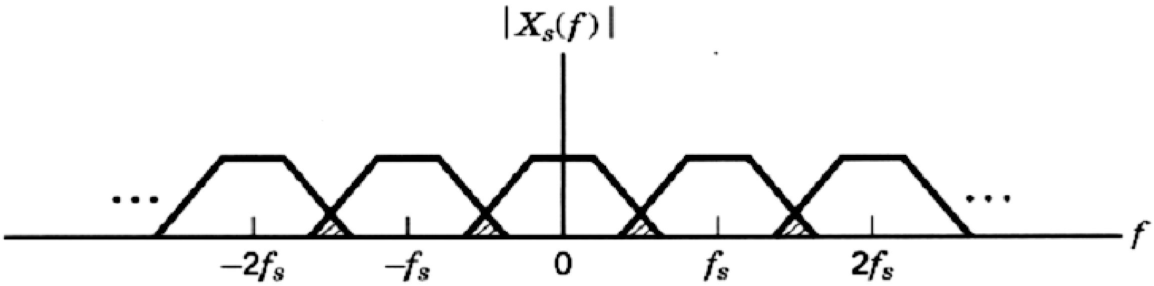
$$X_s(f) = X(f) * X_\delta(f) = X(f) * [\frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - nf_s)] = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

When the sampling rate is chosen $f_s = 2f_m$ each spectral replicate is separated from each of its neighbors by a frequency band exactly equal to f_s hertz, and the analog waveform ca theoretically be completely recovered from the samples, by the use of filtering. It should be clear that if $f_s > 2f_m$, the replications will be move farther apart in frequency making it easier to perform the filtering operation.

When the sampling rate is reduced, such that $f_s < 2f_m$, the replications will overlap, as shown in figure below, and some information will be lost. This phenomenon is called aliasing.



Sampled spectrum $f_s > 2f_m$



Sampled spectrum $f_s < 2f_m$

A bandlimited signal having no spectral components above f_m hertz can be determined uniquely by values sampled at uniform intervals of $T_s \leq \frac{1}{2f_m} sec.$

The sampling rate is $f_s = \frac{1}{T_s}$

So that $f_s \geq 2f_m$. The sampling rate $f_s = 2f_m$ is called Nyquist rate.

Example: Find the Nyquist rate and Nyquist interval for the following signals.

i- $m(t) = \frac{\sin(500\pi t)}{\pi t}$

ii- $m(t) = \frac{1}{2\pi}\cos(4000\pi t)\cos(1000\pi t)$

Solution:

i- $wt = 500\pi t \qquad \therefore 2\pi f = 500\pi \qquad \rightarrow f = 250Hz$

Nyquist interval $= \frac{1}{2f_{max}} = \frac{1}{2\times 250} = 2\,msec.$

Nyquist rate $=2f_{max} = 2 \times 250 = 500Hz$

ii- $m(t) = \frac{1}{2\pi}\left[\frac{1}{2}\{\cos(4000\pi t - 1000\pi t) + \cos(4000\pi t + 1000\pi t)\}\right]$

$$= \frac{1}{4\pi}\{\cos(3000\pi t) + \cos(5000\pi t)\}$$

Then the highest frequency is 2500Hz

Nyquist interval $= \frac{1}{2f_{max}} = \frac{1}{2\times 2500} = 0.2\,msec.$

Nyquist rate $=2f_{max} = 2 \times 2500 = 5000Hz$

H. W:

Find the Nyquist interval and Nyquist rate for the following:

i- $\frac{1}{2\pi}\cos(400\pi t)\,.\cos(200\pi t)$

ii- $\frac{1}{\pi}sin\pi t$

Example:

A waveform [20+20sin(500t+30°)] is to be sampled periodically and reproduced from these sample values. Find maximum allowable time interval between sample values, how many sample values are needed to be stored in order to reproduce 1 sec of this waveform?.

Solution:

$$x(t) = 20 + 20\sin(500t + 30^0)$$

$$w = 500 \rightarrow 2\pi f = 500 \rightarrow f = 79.58\ Hz$$

Minimum sampling rate will be twice of the signal frequency:

$$f_{s(\min)} = 2 \times 79.58 = 159.15\ Hz$$

$$T_{s(max)} = \frac{1}{f_{s(\min)}} = \frac{1}{159.15} = 6.283\ msec.$$

Number of sample in $1sec = \frac{1}{6.283msec} = 159.16 \approx 160\ sample$