Lecture 7

1. Second-Order Circuit

In this lecture we will consider circuits containing two storage elements. These are known as *second-order* circuits because their responses are described by differential equations that contain second derivatives.

Typical examples of second-order circuits are *RLC* circuits, in which the three kinds of passive elements are present. Examples of such circuits are shown in Fig. 1(a) and (b). Other examples are *RL* and *RC* circuits, as shown in Fig. 1(c) and (d). It is apparent from Fig. 1 that a second-order circuit may have two storage elements of different type or the same type (provided elements of the same type cannot be represented by an equivalent single element). An op amp circuit with two storage elements may also be a second-order circuit. As with first-order circuits, a second-order circuit may contain several resistors and dependent and independent sources.

> A **second-order circuit** is characterized by a second-order differential equation. It consists of resistors and the equivalent of two energy storage elements.

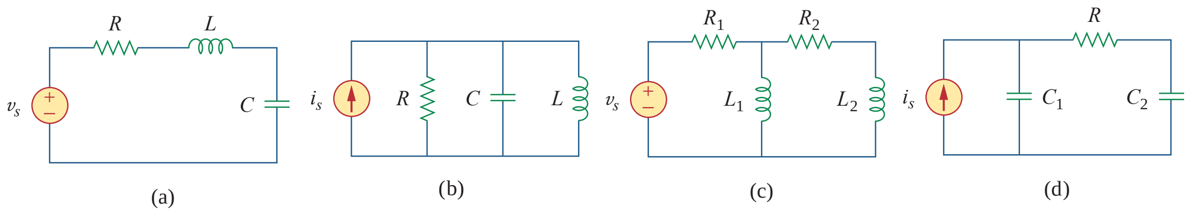


Figure 1 Typical examples of second-order circuits:(a) series *RLC* circuit, (b) parallel *RLC* circuit, (c) *RL* circuit, (d) *RC* circuit.

We will first consider circuits that are excited by the initial conditions of the storage elements. Although these circuits may contain dependent sources, they are free of independent sources. These source-free circuits will give natural responses as expected. Later we will consider circuits that are excited by independent sources. These circuits will give both the transient response and the steady-state response. We consider only dc independent sources in this lecture.

We begin by learning how to obtain the initial conditions for the circuit variables and their derivatives, as this is crucial to analyzing second-order circuits. Then we consider series and parallel *RLC* circuits such as shown in Fig. 1 for the two cases of excitation: by initial conditions of the energy storage elements and by step inputs.

2 Finding Initial and Final Values

Students are usually comfortable getting the initial and final values of v and i but often have difficulty finding the initial values of their derivatives:

dv/dt and di/dt. For this reason, this section is explicitly devoted to the subtleties of getting $v(0)$, $i(0)$, $dv(0)/dt$, $di(0)/dt$, $i(\infty)$, and $v(\infty)$. Unless otherwise stated in this lecture, v denotes capacitor voltage, while i is the inductor current.

There are two key points to keep in mind in determining the initial conditions.

First—as always in circuit analysis—we must carefully handle the polarity of voltage $v(t)$ across the capacitor and the direction of the current $i(t)$ through the inductor. Keep in mind that v and i are defined strictly according to the passive sign convention. One should carefully observe how these are defined and apply them accordingly.

Second, keep in mind that the capacitor voltage is always continuous so that

$$v(0^+) = v(0^-) \tag{1}$$

and the inductor current is always continuous so that

$$i(0^+) = i(0^-) \tag{2}$$

where

$t = 0^-$ denotes the time just before a switching event and $t = 0^+$ is the time just after the switching event, assuming that the switching event takes place at $t = 0$.

Thus, in finding initial conditions, we first focus on those variables that cannot change abruptly, capacitor voltage and inductor current, by applying Eq. (1). The following examples illustrate these ideas.

Example 1 The switch in Fig. 2 has been closed for a long time. It is open at $t = 0$. Find: (a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)/dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$.

Solution:

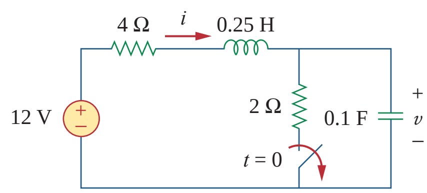


Figure 2 For Example 1.

(a) If the switch is closed a long time before $t = 0$, it means that the circuit has reached dc steady state at $t = 0$. At dc steady state, the inductor acts like a short circuit, while the capacitor acts like an open circuit, so we have the circuit in Fig. 3(a) at $t = 0^-$. Thus,

$$i(0^-) = \frac{12}{4 + 2} = 2\text{ A}, \qquad v(0^-) = 2i(0^-) = 4\text{ V}$$

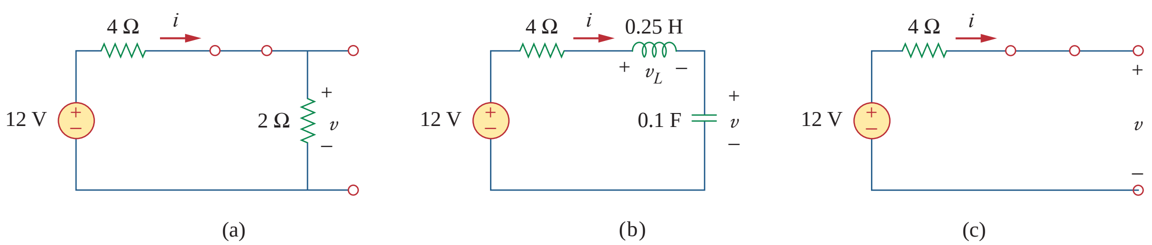


Figure 3 Equivalent circuit of that in Fig. 2 for: (a) $t = 0^-$, (b) $t = 0^+$, (c) $t \to \infty$.

As the inductor current and the capacitor voltage cannot change abruptly,

$$i(0^+) = i(0^-) = 2\text{ A}, \qquad v(0^+) = v(0^-) = 4\text{ V}$$

(b) At $t = 0^+$, the switch is open; the equivalent circuit is as shown in Fig. 3(b). The

same current flows through both the inductor and capacitor. Hence,

$$i_C(0^+) = i(0^+) = 2\text{ A}$$

Since $C\,dv/dt = i_C$, $dv/dt = i_C/C$, and

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{2}{0.1} = 20\text{ V/s}$$

Similarly, since $L\,di/dt = v_L$, $di/dt = v_L/L$. We now obtain v_L by applying KVL to the loop in Fig. 3(b).

$$-12 + 4i(0^+) + v_L(0^+) + v(0^+) = 0$$

or

$$v_L(0^+) = 12 - 8 - 4 = 0$$

Thus,

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{0.25} = 0\text{ A/s}$$

(c) For $t > 0$, the circuit undergoes transience. But as $t \to \infty$, the circuit reaches steady state again. The inductor acts like a short circuit and the capacitor like an open circuit, so that the circuit in Fig. 3(b) becomes that shown in Fig. 3(c), from which we have

$$i(\infty) = 0\text{ A}, \qquad v(\infty) = 12\text{ V}$$

Example 2 In the circuit of Fig. 4, calculate: (a) $i_L(0^+)$, $v_C(0^+)$, $v_R(0^+)$, (b) $di_L(0^+)/dt$, $dv_C(0^+)/dt$, $dv_R(0^+)/dt$, (c) $i_L(\infty)$, $v_C(\infty)$, $v_R(\infty)$.

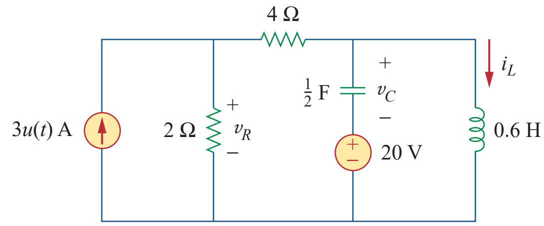


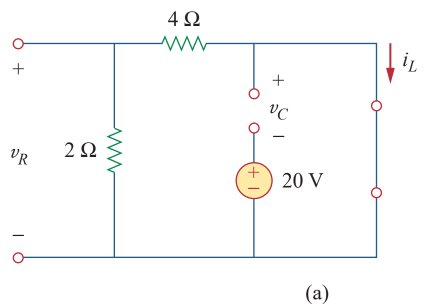
Figure 4 For Example 2.

Solution:

(a) For $t < 0$, $3u(t) = 0$. At $t = 0^-$, since the circuit has reached steady state, the inductor can be replaced by a short circuit, while the capacitor is replaced by an open circuit as shown in Fig. 5(a). From this figure we obtain

$$i_L(0^-) = 0, \qquad v_R(0^-) = 0, \qquad v_C(0^-) = -20 \text{ V} \tag{2.1}$$

Although the derivatives of these quantities at $t = 0^-$ are not required, it is evident that they are all zero, it is evident that they are all zero, since the circuit has reached steady state and nothing changes.



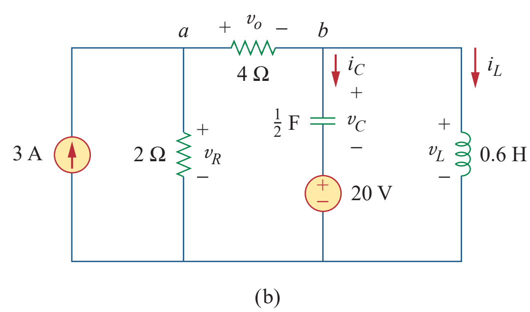


Figure 5 The circuit in Fig. 4 for: (a) $t = 0^-$, (b) $t = 0^+$.

For $t > 0$, $3u(t) = 3$, so that the circuit is now equivalent to that in Fig. 5(b). Since the inductor current and capacitor voltage cannot change abruptly,

$$i_L(0^+) = i_L(0^-) = 0, \qquad v_C(0^+) = v_C(0^-) = -20 \text{ V} \tag{2.2}$$

Although the voltage across the 4-Ω resistor is not required, we will use it to apply KVL and KCL; let it be called v_o. Applying KCL at node a in Fig. 5(b) gives

$$3 = \frac{v_R(0^+)}{2} + \frac{v_o(0^+)}{4} \tag{2.3}$$

Applying KVL to the middle mesh in Fig. 5(b) yields

$$-v_R(0^+) + v_o(0^+) + v_C(0^+) + 20 = 0 \tag{2.4}$$

Since $v_C(0^+) = -20$ V from Eq. (2.2), Eq. (2.4) implies that

$$v_R(0^+) = v_o(0^+) \tag{2.5}$$

From Eqs. (2.3) and (2.5), we obtain

$$v_R(0^+) = v_o(0^+) = 4 \text{ V} \tag{2.6}$$

(b) Since $L\, di_L/dt = v_L$,

$$\frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L}$$

But applying KVL to the right mesh in Fig. 5(b) gives

$$v_L(0^+) = v_C(0^+) + 20 = 0$$

Hence,

$$\frac{di_L(0^+)}{dt} = 0 \tag{2.7}$$

Similarly, since $C\, dv_C/dt = i_C$, then $dv_C/dt = i_C/C$. We apply KCL at node b in Fig. 5(b) to get i_C

$$\frac{v_o(0^+)}{4} = i_C(0^+) + i_L(0^+) \tag{2.8}$$

Since $v_o(0^+) = 4$ and $i_L(0^+) = 0$, $i_C(0^+) = 4/4 = 1$ A. Then

$$\frac{dv_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{1}{0.5} = 2 \text{ V/s} \tag{2.9}$$

To get $dv_R(0^+)/dt$, we apply KCL to node a and obtain

$$3 = \frac{v_R}{2} + \frac{v_o}{4}$$

Taking the derivative of each term and setting $t = 0^+$ gives

$$0 = 2\frac{dv_R(0^+)}{dt} + \frac{dv_o(0^+)}{dt} \tag{2.10}$$

We also apply KVL to the middle mesh in Fig. 5(b) and obtain

$$-v_R + v_C + 20 + v_o = 0$$

Again, taking the derivative of each term and setting $t = 0^+$ yields

$$-\frac{dv_R(0^+)}{dt} + \frac{dv_C(0^+)}{dt} + \frac{dv_o(0^+)}{dt} = 0$$

Substituting for $dv_C(0^+)/dt = 2$ gives

$$\frac{dv_R(0^+)}{dt} = 2 + \frac{dv_o(0^+)}{dt} \tag{2.11}$$

From Eqs. (2.10) and (2.11), we get

$$\frac{dv_R(0^+)}{dt} = \frac{2}{3} \text{ V/s}$$

We can find $di_R(0^+)/dt$ although it is not required. Since $v_R = 5i_R$,

$$\frac{di_R(0^+)}{dt} = \frac{1}{5}\frac{dv_R(0^+)}{dt} = \frac{1}{5}\frac{2}{3} = \frac{2}{15}\text{ A/s}$$

(c) As $t \to \infty$, the circuit reaches steady state. We have the equivalent circuit in Fig. 5(a) except that the 3-A current source is now operative. By current division principle,

$$i_L(\infty) = \frac{2}{2+4}3\text{ A} = 1\text{ A}$$

$$v_R(\infty) = \frac{4}{2+4}3\text{ A} \times 2 = 4\text{ V}, \qquad v_C(\infty) = -20\text{ V}$$

Practice Problems

Problem 1

For the circuit in Fig. P1, find: (a) $i_L(0^+), v_C(0^+), v_R(0^+)$, (b) $di_L(0^+)/dt, dv_C(0^+)/dt, dv_R(0^+)/dt$, (c) $i_L(\infty), v_C(\infty), v_R(\infty)$.

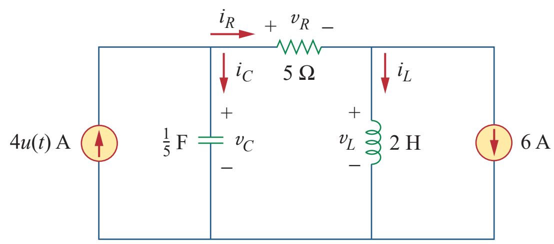


Figure P1

Answer: (a) −6 A, 0, 0, (b) 0, 20 V/s, 0, (c) −2 A, 20 V, 20 V.

Problem 2

The switch in Fig. P2 was open for a long time but closed at $t = 0$. Determine: (a) $i(0^+), v(0^+)$, (b) $di(0^+)/dt, dv(0^+)/dt$, (c) $i(\infty), v(\infty)$.

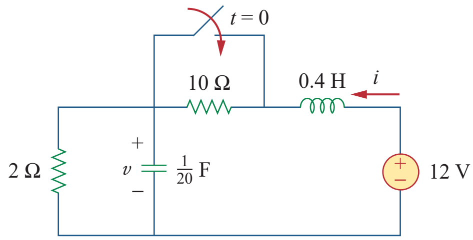


Figure P2

Answer: (a) 1 A, 2 V, (b) 25 A/s, 0 V/s, (c) 6 A, 12 V.