Lecture 11

1. General Second-Order Circuits

Now that we have mastered series and parallel *RLC* circuits, we are prepared to apply the ideas to any second-order circuit having one or more independent sources with constant values. Although the series and parallel *RLC* circuits are the second-order circuits of greatest interest, other second-order circuits including op amps are also useful.
Given a second-order circuit, we determine its step response $x(t)$ (which may be voltage or current) by taking the following four steps:

1. We first determine the initial conditions $x(0)$ and $dx(0)/dt$ and the final value $x(\infty)$, as discussed in Lecture 7.

2. We turn off the independent sources and find the form of the transient response $x_t(t)$ by applying KCL and KVL.

   Once a second-order differential equation is obtained, we determine its characteristic roots. Depending on whether the response is overdamped, critically damped, or underdamped, we obtain $x_t(t)$ with two unknown con-stants as we did in the previous sections.

3. We obtain the steady-state response as

$$x_{ss}(t) = x(\infty) \tag{1}$$

   where $x(\infty)$ is the final value of x, obtained in step 1.

4. The total response is now found as the sum of the transient response and steady-state response

$$x(t) = x_t(t) + x_{ss}(t) \tag{2}$$

We finally determine the constants associated with the transient response by imposing the initial conditions $x(0)$ and $dx(0)/dt$, determined in step 1.

We can apply this general procedure to find the step response of any second-order circuit, including those with op amps. The following examples illustrate the four steps.

Example 1 Find the complete response v and then i for $t > 0$ in the circuit of Fig. 1.

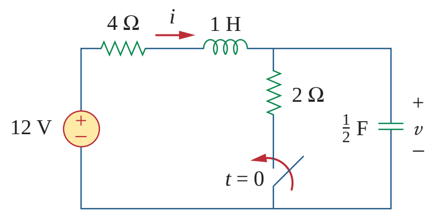

Figure 1 For Example 1

Solution:

We first find the initial and final values. At $t = 0^-$, the circuit is at steady state. The switch is open; the equivalent circuit is shown in Fig. 2(a). It is evident from the figure that

$$v(0^-) = 12 \text{ V}, \qquad i(0^-) = 0$$

At $t = 0^+$, the switch is closed; the equivalent circuit is in Fig. 2(b). By the continuity of capacitor voltage and inductor current, we know that

$$v(0^+) = v(0^-) = 12 \text{ V}, \qquad i(0^+) = i(0^-) = 0 \tag{1.1}$$

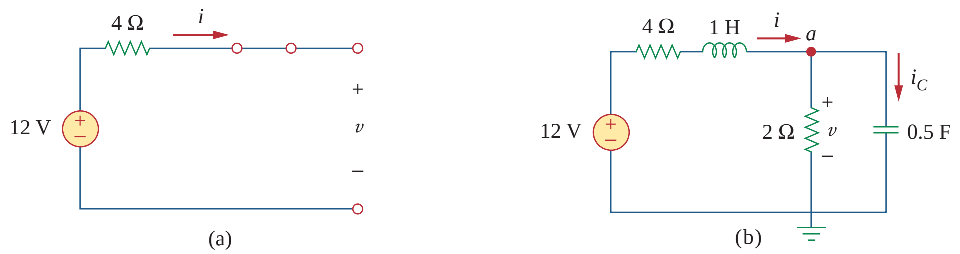

Figure 2 Equivalent circuit of the circuit in Fig. 1 for: (a) $t < 0$, (b) $t > 0$.

To get $dv(0^+)/dt$, we use $C\,dv/dt = i_C$ or $dv/dt = i_C/C$. Applying KCL at node a in Fig. 2(b),

$$i(0^+) = i_C(0^+) + \frac{v(0^+)}{2}$$

$$0 = i_C(0^+) + \frac{12}{2} \qquad \Rightarrow \qquad i_C(0^+) = -6 \text{ A}$$

Hence,

$$\frac{dv(0^+)}{dt} = \frac{-6}{0.5} = -12 \text{ V/s} \tag{1.2}$$

The final values are obtained when the inductor is replaced by a short circuit and the capacitor by an open circuit in Fig.2(b), giving

$$i(\infty) = \frac{12}{4 + 2} = 2 \text{ A}, \qquad v(\infty) = 2i(\infty) = 4 \text{ V} \tag{1.3}$$

Next, we obtain the form of the transient response for $t > 0$. By turning off the 12-V voltage source, we have the circuit in Fig. 3. Applying KCL at node a in Fig. 3 gives

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} \tag{1.4}$$

Applying KVL to the left mesh results in

$$4i + 1\frac{di}{dt} + v = 0 \tag{1.5}$$

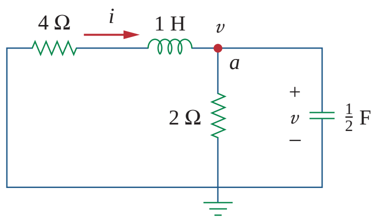

Figure 3 Obtaining the form of the transient response for Example 1.

Since we are interested in v for the moment, we substitute i from Eq. (1.4) into Eq. (1.5). We obtain

$$2v + 2\frac{dv}{dt} + \frac{1}{2}\frac{dv}{dt} + \frac{1}{2}\frac{d^2v}{dt^2} + v = 0$$

or

$$\frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 0$$

From this, we obtain the characteristic equation as

$$s^2 + 5s + 6 = 0$$

with roots $s = -2$ and $s = -3$. Thus, the natural response is

$$v_n(t) = Ae^{-2t} + Be^{-3t} \tag{1.6}$$

where A and B are unknown constants to be determined later. The steady-state response is

$$v_{ss}(t) = v(\infty) = 4 \tag{1.7}$$

The complete response is

$$v(t) = v_t + v_{ss} = 4 + Ae^{-2t} + Be^{-3t} \tag{1.8}$$

We now determine A and B using the initial values. From Eq. (1.1), $v(0) = 12$. Substituting this into Eq. (1.8) at $t = 0$ gives

$$12 = 4 + A + B \quad \Rightarrow \quad A + B = 8 \tag{1.9}$$

Taking the derivative of v in Eq. (1.8),

$$\frac{dv}{dt} = -2Ae^{-2t} - 3Be^{-3t} \tag{1.10}$$

Substituting Eq. (1.2) into Eq. (1.10) at $t = 0$ gives

$$-12 = -2A - 3B \quad \Rightarrow 2A + 3B = 12 \tag{1.11}$$

From Eqs. (1.9) and (1.11), we obtain

$$A = 12, \quad B = -4$$

so that Eq. (1.8) becomes

$$v(t) = 4 + 12e^{-2t} - 4e^{-3t}\text{ V}, \quad t > 0 \tag{1.12}$$

From v, we can obtain other quantities of interest by referring to Fig. 2(b). To obtain i, for example,

$$\begin{aligned} i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} &= 2 + 6e^{-2t} - 2e^{-3t} - 12e^{-2t} + 6e^{-3t} \\ &= 2 - 6e^{-2t} + 4e^{-3t}\text{ A}, \quad t > 0 \end{aligned} \tag{1.13}$$

Notice that $i(0) = 0$, in agreement with Eq. (1.1).

Practice Problems

Problem 1

Determine v and i for $t > 0$ in the circuit of Fig. P1.

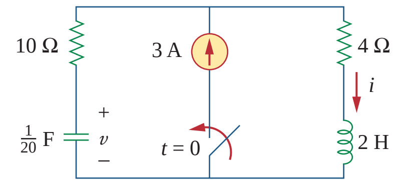

Figure P1

Answer: $12(1 - e^{-5t})$ V, $3(1 - e^{-5t})$ A.

Problem 2

For $t > 0$, obtain $v_o(t)$ in the circuit of Fig.P2. (*Hint:* First find v_1 and v_2.)

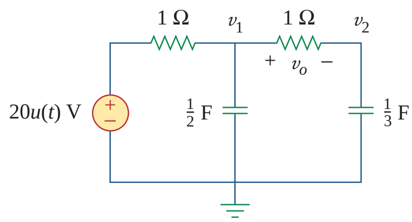

Figure P2

Answer: $8(e^{-t} - e^{-6t})$ V, $t > 0$.

Problem 3

Find $v_o(t)$ for $t > 0$ in the circuit of Fig. P3.

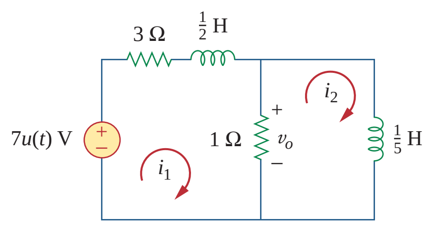

Figure P3