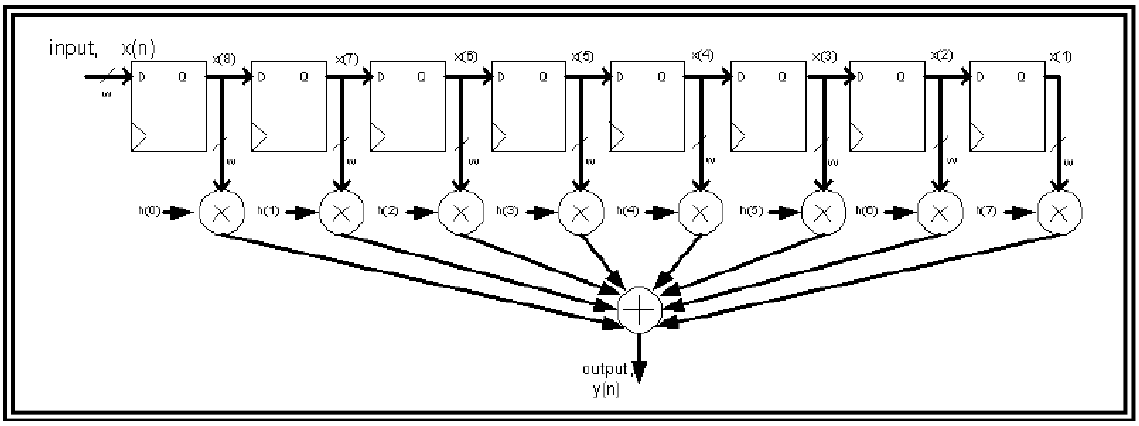
Ministry of Higher Education and Scientific Research
Middle Technical University
Electrical Engineering Technical College

Training package in

Digital Signal Processing (Conventional Solution of Difference Equation)

For

Students of third class
Control and Automation Engineering Techniques



By

Prof. Dr. Raaed F. Hassan
Dep. Control and Automation Eng. Tech.
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Conventional Solution of LTI Difference Equations

1/ Overview

1 / A –Target population :-

For students of third class

Department of Medical Instrumentation Eng. Techniques

1 / B –Rationale :-

This unit introduces conventional method for solving LTI difference equations

1 / C –Central Idea :-

The major topics discussed in this unit are included in the following outline.

- Solution of Linear Constant – Coefficient Difference Equations
- The homogeneous solution of a difference equation.
- The particular solution of the difference equation
- The total Solution of the difference equation.

2/ Performance Objectives :-

After studying the 5th modular unit , the student will be able to:-

1. Determine the homogeneous solution of a difference equation.
2. Determine the particular solution of the difference equation
3. Determine the total Solution of the difference equation.

3/ Pre test :-

Circle the correct answer :-

1. LTI is the abbreviation of :-

a- Non linear time varying systems.

b- Linear time varying systems

c- Linear time invariant systems

d- Non of the above.

2. LTI systems can be represented using:

a- Mathematical model or block diagram

b- Digital numbers

c- Unit impulse signal

d- Unit step signal

3. Difference equation represents:-

a- Output equation.

b- Block diagram model of LTI systems.

c- Mathematical model of LTI systems.

d- Input equation.

4/ the text :-

Finite – Duration and Infinite – Duration Impulse response

It is convenient, however, to subdivide the class of linear – time invariant system into two types, those that have a finite – duration impulse response (FIR) and those that have an infinite – duration impulse response (IIR).

Thus an FIR system has an impulse response that is zero outside of some finite time interval.

The convolution formula for such a system reduces to:-

$$y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k)$$

In effect, the system acts as a window that views only the most M input signal samples in forming the output. Thus we say that an FIR system has a finite memory of length M samples.

In contrast, an IIR linear – time invariant system has an infinite duration impulse response. Its output, based on the convolution formula, is

$$y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k)$$

Recursive and Nonrecursive Discrete – time systems.

There are many systems where it is either necessary or desirable to express the output of the system not only in terms of the present and past

values of the input, but also in terms of the already available past output values.
Suppose that we wish to compute the cumulative average of a signal *x(n)* in the interval $0 \leq k \leq n$, defined as

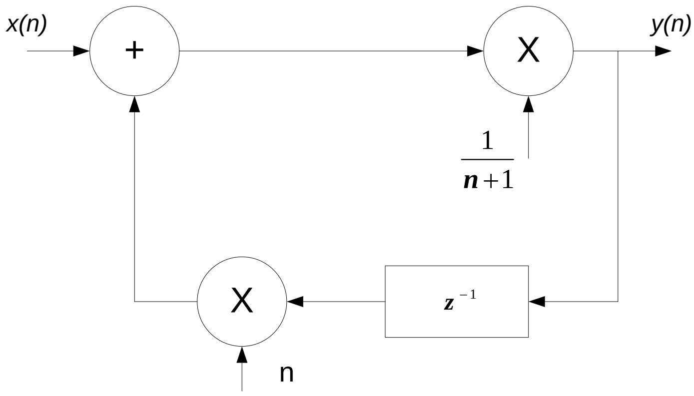
$$y(n) = \frac{1}{n+1}\sum_{k=0}^{n} x(k)$$

The computation of y(n) requires the storage of all the input samples *x(k)* for $0 \leq k \leq n$. Since n is increasing, our memory requirements grow linearly with time.
Our intuition suggests, however, that *y(n)* can be computed more efficiently by utilizing the previous output value *y(n-1)*.

$$(n+1)y(n) = \sum_{k=0}^{n-1} x(k) + x(n)$$

$$= ny(n-1) + x(n)$$

$$y(n) = \frac{n}{n+1}y(n-1) + \frac{1}{n+1}x(n)$$



<u>Square – Root Algorithm</u>
Many computers and calculators compute the square root of a positive number A using the iterative algorithm.

$$s_n = \frac{1}{2}\left(s_{n-1} + \frac{A}{s_{n-1}}\right), \qquad n = 0,1, \dots \dots$$

Where Sn-1 is an initial guess (estimate) of $\sqrt{A}$. As the iteration converges we have $s_n \cong s_{n-1}$. Then it easily follows that $s_n \cong \sqrt{A}$.

Solution of Linear Constant – Coefficient Difference Equations

Given a linear constant – coefficient difference equation as the input – output relationship describing a linear time – invariant system, our objective in this subsection is to determine an explicit expression for the output *y(n)*. The method that is developed is termed the direct method.

Basically, the goal is to determine the output $y(n), n \geq 0$, of the system given a specific input $x(n),\ n \geq 0$, and a set of initial conditions. The direct solution method assumes that the total solution is the sum of two parts:

$$y(n) = y_h(n) + y_p(n)$$

The part $y_h(n)$ is known as the homogeneous or complementary solution, whereas $y_p(n)$ is called the particular solution.

The homogeneous solution of a difference equation.

We begin the problem of solving the linear constant – coefficient difference equation given by

$$\sum_{\boldsymbol{k=0}}^{\boldsymbol{N}} \boldsymbol{a_k y(n-k)} = \sum_{\boldsymbol{k=0}}^{\boldsymbol{M}} \boldsymbol{b_k x(n-k)}$$

By assuming that the input $x(n) = 0$. Thus we will first obtain the solution to the homogeneous difference equation

$$\sum_{k=0}^{N} a_k y(n-k) = 0$$

Basically, we assume that the solution is in the form of an exponential, that is.

$$y_h(n) = \lambda^n$$

Where the subscript *h* on *y(n)* is used to denote the solution to the homogeneous difference equation. If we substitute this assumed solution, we obtain the polynomial equation

$$\sum_{k=0}^{N} a_k \lambda^{n-k} = 0$$

Or

$$\lambda^{n-N}(\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_{N-1}\lambda + a_N) = 0$$

The polynomial in parentheses is called the characteristic polynomial of the system. In general, it has N roots, which we denote as $\lambda_1, \lambda_2, \ldots \lambda_N$. The roots can be real or complex valued. In practice the coefficients a_1, $a_2, \ldots a_N$ are usually real.

For the moment, let us assume that the roots are distinct, that is, there are no multiple – order roots. Then the most general solution to the homogeneous difference equation is

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + \cdots + C_N\lambda_N^n$$

Where $C_1, C_2 \ldots C_N$ are weighting coefficients.

Example

Determine the homogeneous solution of the system described by the first – order difference equation

$$y(n) + a_1 y(n-1) = x(n)$$

Solution

The assumed solution obtained by setting *x(n) = 0* is

$$y_h(n) = \lambda^n$$

When we substitute this solution in the above equation with *x(n) = 0*, we obtain

$$\lambda^n + a_1 \lambda^{n-1} = 0$$
$$\lambda^{n-1}(\lambda + a_1) = 0$$
$$\lambda = -a_1$$

Therefore, the solution to the homogeneous difference equation is

$$y_h(n) = C\lambda^n = C(-a_1)^n$$

The zero – input response of the system is

$$y(0) = -a_1 y(-1)$$

Or

$$y_h(0) = C$$

And hence the zero – input response of the system is

$$y_h(n) = (-a_1)^{n+1} y(-1) \qquad n \geq 0$$

Example

Determine the zero – input response of the system described by the homogeneous second – order difference equation

$$y(n) - 3y(n-1) - 4y(n-2) = 0$$

Solution

First we determine the solution to the homogeneous equation. We assume the solution to be the exponential

$$y_h(n) = \lambda^n$$

Upon substitution of this solution, we obtain the characteristic equation

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0$$
$$\lambda^{n-2}(\lambda^2 - 3\lambda - 4) = 0$$

Therefore, the roots are $\lambda = -1, 4$, and the general form of the solution to the homogeneous equation is

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n$$
$$= C_1(-1)^n + C_2(4)^n$$

The zero – input response of the system can be obtained from the homogenous solution by evaluating the constants, given the initial conditions *y(-1)* and *y(-2)*. From the difference equation we have

$$y(0) = 3y(-1) + 4y(-2)$$
$$y(1) = 3y(0) + 4y(-1)$$
$$= 3[3y(-1) + 4y(-2)] + 4y(-1)$$
$$= 13y(-1) + 12y(-2)$$

On the other hand

$$y(0) = C_1 + C_2$$
$$y(1) = -C_1 + 4C_2$$

By equating these two sets of relations, we have

$$C_1 + C_2 = 3y(-1) + 4y(-2)$$
$$-C_1 + 4C_2 = 13y(-1) + 12y(-2)$$

The solution of these two equations is

$$C_1 = -\frac{1}{5}y(-1) + \frac{4}{5}y(-2)$$
$$C_2 = \frac{16}{5}y(-1) + \frac{16}{5}y(-2)$$

Therefore, the zero – input response of the system is

$$y_h(n) = \left[-\frac{1}{5}y(-1) + \frac{4}{5}y(-2)\right](-1)^n + \left[\frac{16}{5}y(-1) + \frac{16}{5}y(-2)\right](4)^n \qquad n \geq 0$$

If λ_1 is a root of multiplicity *m* then the homogenous equation becomes

$$y_h(n) = C_1\lambda_1^n + C_2 n\lambda_1^n + C_3 n^2\lambda_1^n + \cdots + C_m n^{m-1}\lambda_1^n + C_{m+1}\lambda_{m+1}^n + \cdots + C_N\lambda_n$$

The particular solution of the difference equation

The particular solution $y_p(n)$ is required to satisfy the difference equation for the specific input signal $x(n), n \geq 0$. In other words, $y_p(n)$ is any solution satisfying

$$\sum_{k=0}^{N} a_k y_p(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

To solve above equation, we assume for $y_p(n)$, a form that depends on the form of the input *x(n)*.

Example

Determine the particular solution of the first order difference equation

$$y(n) + a_1 y(n-1) = x(n) \qquad |a_1| < 1$$

When the input $x(n) = u(n)$.

Solution

Since the input sequence $x(n)$ is a constant for $n \geq 0$, the form of the solution that we assume is also a constant. Hence the assumed solution of the difference equation to the forcing function $x(n)$, called the particular solution of the difference equation, is

$$y_p(n) = Ku(n)$$

Where K is a scale factor determined so that the difference equation is satisfied.

$$Ku(n) + a_1Ku(n-1) = u(n)$$

To determine K, we must evaluate this equation for any $n \geq 1$. Where none of the terms vanish. Thus

$$K + a_1K = 1 \qquad \rightarrow K = \frac{1}{1+a_1}$$

Therefore, the particular solution to the difference equation is

$$y_p(n) = \frac{1}{1+a_1}u(n)$$

In this example, the input $x(n)$, $n \geq 0$, is a constant and the form assumed for the particular solution is also constant. If $x(n)$ is an exponential, we would assume that the particular solution is also be a sinusoid. The table below provides the general form of the particular solution for several types of excitation.

Input Signal	Particular Solution
$x(n)$	$y_p(n)$
AM^n	KM^n
An^M	$K_0n^M+K_1n^{M-1}+\ldots+K_M$
A^nn^M	$A^n(K_0n^M+K_1n^{M-1}+\ldots+K_M)$
$\begin{Bmatrix} Acos\omega_0 n \\ Asin\omega_0 n \end{Bmatrix}$	$K_1cos\omega_0 n + K_2sin\omega_0 n$

The total Solution of the difference equation.

The linearity property of the linear constant – coefficient difference equation allows us to add the homogeneous solution and the particular solution in order to obtain the total solution. Thus

$$y(n) = y_h(n) + y_p(n)$$

The resultant sum *y(n)* contains the constant parameters $\{C_i\}$ embodied in the homogeneous solution component $y_h(n)$. These constants can be determined to satisfy the initial conditions.

Example

Determine the total solution *y(n), n ≥ 0,* to the difference equation

$$y(n) + a_1 y(n-1) = x(n) \qquad (1)$$

When *x(n)* is a unit step sequence.

Solution

The homogeneous solution is

$$y_h(n) = C(-a_1)^n$$

The particular solution is

$$y_p(n) = \frac{1}{1 + a_1}$$

The total solution is

$$y(n) = C(-a_1)^n + \frac{1}{1 + a_1} \qquad n \geq 0 \qquad (2)$$

Where the constant C is determined to satisfy the initial condition *y(-1).* In particular, suppose that we wish to obtain the zero – state response of the system described by the first – order difference equation. Then we set *y(-1)=0*. To evaluate *C*, we evaluate (1) at *n = 0* obtaining

$$y(0) + a_1 y(-1) = 1$$

$$y(0) = 1$$

$$y(0) = C + \frac{1}{1 + a_1}$$

$$C = \frac{a_1}{1 + a_1}$$

Substitution for C into (2) yields the zero state response of the system

$$y_{zs}(n) = \frac{1 - (-a_1)^{n+1}}{1 + a_1}$$

If we evaluate the parameter *C* in (2) under the condition that *y(-1)≠0,* the total solution will include the zero – input response as well as the zero – state response of the system. In this case (1) yields

$$y(0) + a_1 y(-1) = 1$$

$$y(0) = -a_1 y(-1) + 1$$

On the other hand, (2) yields

$$y(0) = C + \frac{1}{1+a_1}$$

By equating these two relations, we obtain

$$C + \frac{1}{1+a_1} = -a_1 y(-1) + 1$$

Finally, if we substitute this value of C into (2), we obtain

$$y(n) = (-a_1)^{n+1} y(-1) + \frac{1-(-a_1)^{n+1}}{1+a_1} \qquad n \geq 0$$

$$= y_{zi}(n) + y_{zs}(n)$$

5/ Post test :-

1. Solution of difference equation contains:
   a) One part.
   b) Two parts.
   c) Three parts.
   d) Four parts.
2. In homogenous solution the input signals equal to:
   a) Zero.
   b) Unit step.
   c) Unit impulse.
   d) Exponential.
3. The particular solution is:
   a) System response to specific input.
   b) System response to zero input.
   c) System response to unit impulse.
   d) System response to unit step.
4. The total solution of difference equation is:

a) Difference between homogenous and particular solutions.
b) Summation of homogenous and particular solutions.
c) Multiplication of homogenous and particular solutions.
d) Division of homogenous and particular solutions.

Key answers

Pre test:

1.b **2.a** **3.c**

Post test

1.b **2.a** **3.a** **4.b**

7/References :-

1. Schaum's Outline of Theory and Problems of Digital Signal processing.
2. Digital signal processing, principles, algorithms, and applications by John G. Proakis and Dimitris G. Manolakis.
3. Signal and systems, Alan Oppenheim.