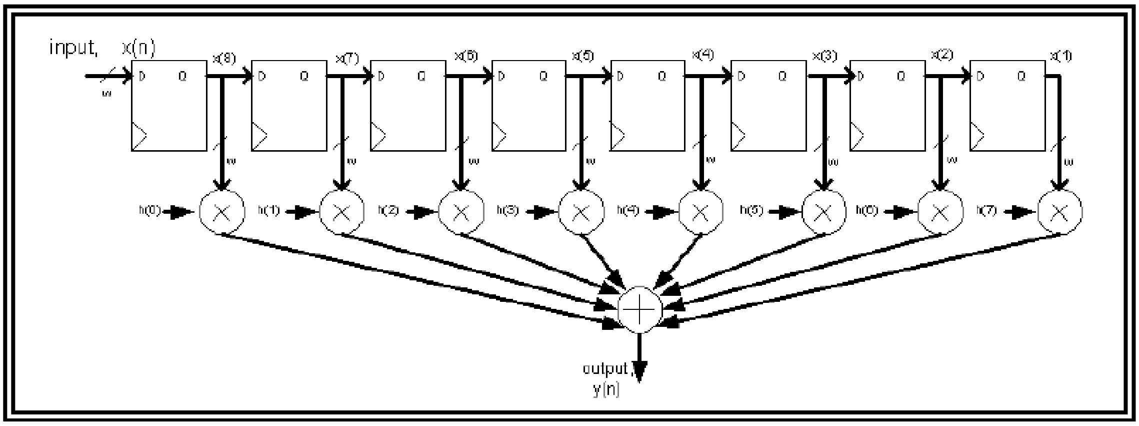
Ministry of Higher Education and Scientific Research
Middle Technical University
Electrical Engineering Technical College

Training package
in

Digital Signal Processing
(Design of IIR Filter)

For
Students of third class
Control and Automation Engineering Techniques



By

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Design of IIR Filter

1/ Overview

1 / A –Target population :-

For students of third class

Department of Medical Instrumentation Eng. Techniques

1 / B –Rationale :-

This unit introduces principles of the infinite impulse response (IIR) filter design and investigates the Bilinear transformation method.

1 / C –Central Idea :-

In this unit, we describe techniques of designing infinite impulse response (IIR) filters. The major topics discussed in this unit are included in the following outline.

- Infinite Impulse Response (IIR) Filter Format
- Bilinear Transformation.

2/ Performance Objectives :-

After studying the 11th modular unit, the student will be able to:-

1. Define IIR filter.
2. Know the Bilinear Transformation.

4/ the text :-

Infinite Impulse Response Filter Design

This lecture investigates a bilinear transformation method for infinite impulse response (IIR) filter design and develops a procedure to design digital Butterworth filters.

Infinite Impulse Response Filter Format

An IIR filter is described using the difference equation

$$y(n) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M) - a_1 y(n-1) - \cdots - a_N y(n-N).$$

The IIR filter transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}},$$

Bilinear Transformation Design Method

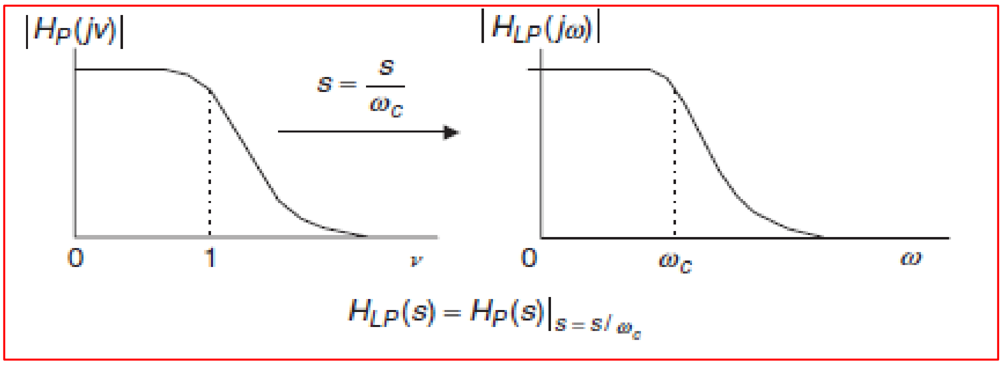
The design procedure includes the following steps:

1. Transforming digital filter specifications into analog filter specifications.
2. Performing analog filter design.
3. Applying bilinear transformation (which will be introduced in the next section) and verifying its frequency response.

Analog Filters Using Low pass Prototype Transformation

This method converts the analog low pass filter with a cutoff frequency of 1 radian per second, called the low pass prototype, into practical analog low pass, high pass, band pass, and band stop filters with their frequency specifications.

Letting $H_p(s)$ be a transfer function of the lowpass prototype, the transformation of the lowpass prototype into a lowpass filter is given in the Figure below.



Let us consider the following first-order lowpass prototype:

$$H_P(s) = \frac{1}{s+1}.$$

Applying the prototype transformation s/ω_c we get an analog lowpass filter with a cutoff frequency of ω_c as

$$H(s) = \frac{1}{s/\omega_c + 1} = \frac{\omega_c}{s+\omega_c}.$$

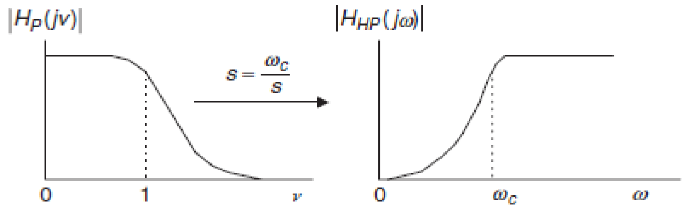
We can obtain the analog frequency response by substituting $s = j\omega$ into above equation, that is,

$$H(j\omega) = \frac{1}{j\omega/\omega_c + 1}.$$

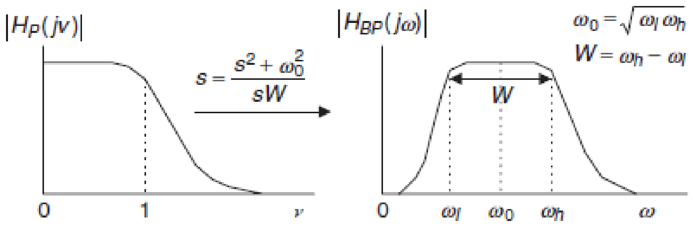
The magnitude response is determined by

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^2}}.$$

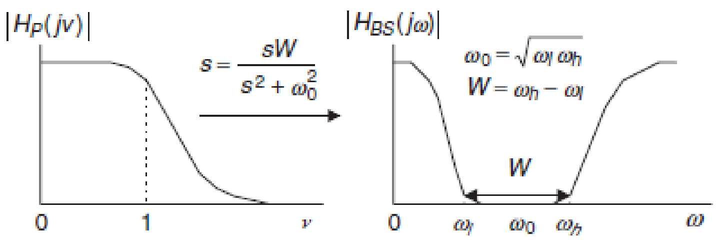
The highpass, bandpass, and bandstop filters using the specified lowpass prototype transformation can be easily verified. We review them in Figures below



$H_{HP}(s) = H_P(s)|_{s=\omega_c/s}$



$H_{BP}(s) = H_P(s)\ \Big|_{s=\frac{s^2+\omega_0^2}{sW}}$



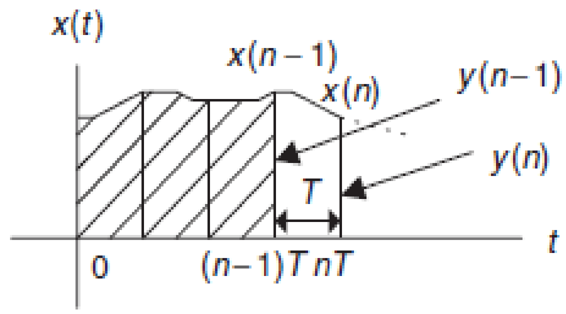
$H_{BS}(s) = H_P(s)\ \Big|_{s=\frac{sW}{s^2+\omega_0^2}}$

Bilinear Transformation and Frequency Warping

In this subsection, we develop the BLT, which converts an analog filter into a digital filter.

We begin by finding the area under a curve using the integration of calculus and the numerical recursive method.

As shown in Figure below,



The area under the curve can be determined using the following integration:

$$y(t) = \int_0^t x(t)dt,$$

where y(t) (area under the curve) and x(t) (curve function) are the output and input of the analog integrator, respectively, and t is the upper limit of the integration.

Applying Laplace transform, we have

$$Y(s) = \frac{X(s)}{s}$$

and find the Laplace transfer function as

$$G(s) = \frac{Y(s)}{X(s)} = \frac{1}{s}.$$

Now we examine the numerical integration method

$$y(n) = y(n-1) + \frac{x(n) + x(n-1)}{2}T,$$

Applying the z-transform on both sides

$$Y(z) = z^{-1}Y(z) + \frac{T}{2}\left(X(z) + z^{-1}X(z)\right).$$

Solving for the ratio Y(z) / X(z), we achieve the z-transfer function as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}.$$

$$\frac{1}{s} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} = \frac{T}{2}\frac{z+1}{z-1}.$$

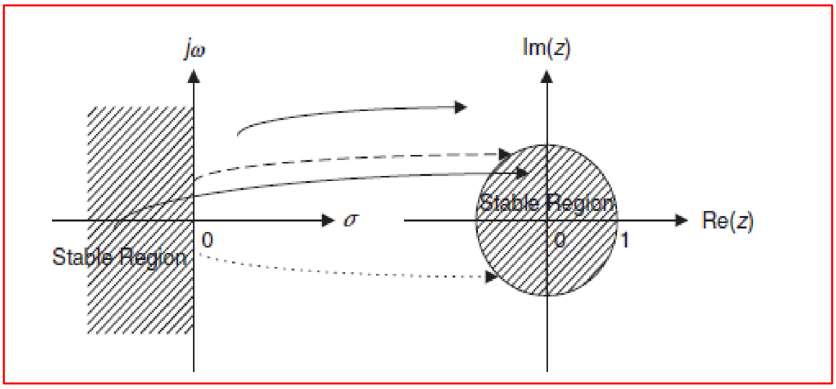
$$s = \frac{2}{T}\frac{z-1}{z+1}.$$

The BLT method is a mapping or transformation of points from the s-plane to the z-plane.

$$z = \frac{1+sT/2}{1-sT/2}.$$

The general mapping properties are summarized as following:

1. The left-half s-plane is mapped onto the inside of the unit circle of the z-plane.
2. The right-half s-plane is mapped onto the outside of the unit circle of the z-plane.



3. The positive jω axis portion in the s-plane is mapped onto the positive half circle (the dashed-line arrow in Figure above) on the unit circle, while the negative jω axis is mapped onto the negative half circle (the dotted line arrow in Figure above) on the unit circle.

Example:

Given an analog filter whose transfer function is

$$H(s) = \frac{10}{s+10},$$

Convert it to the digital filter transfer function and difference equation, respectively, when a sampling period is given as T = 0.01 second.

Solution: Applying the BLT, we have

$$H(z) = H(s)|_{s=\frac{2}{T}\frac{z-1}{z+1}} = \left.\frac{10}{s+10}\right|_{s=\frac{2}{T}\frac{z-1}{z+1}}.$$

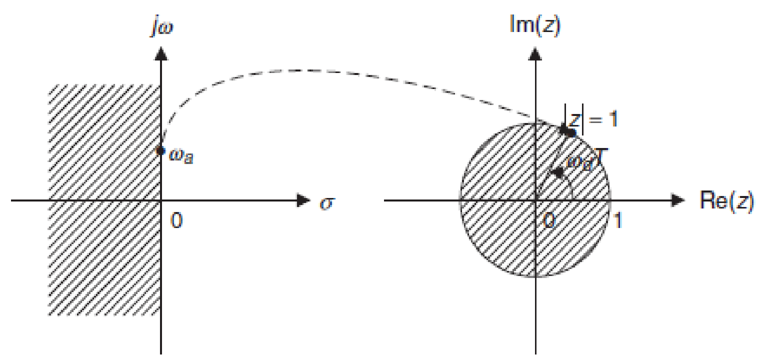
Substituting T = 0.01, it follows that

$$H(z) = \frac{10}{\frac{200(z-1)}{z+1}+10} = \frac{0.05}{\frac{z-1}{z+1}+0.05} = \frac{0.05(z+1)}{z-1+0.05(z+1)} = \frac{0.05z+0.05}{1.05z-0.95}.$$

$$H(z) = \frac{(0.05z+0.05)/(1.05z)}{(1.05z-0.95)/(1.05z)} = \frac{0.0476+0.0476z^{-1}}{1-0.9048z^{-1}}.$$

$$y(n) = 0.0476x(n) + 0.0476x(n-1) + 0.9048y(n-1).$$

Next, we examine frequency mapping between the s-plane and the z-plane. As illustrated in Figure



The analog frequency ωa is marked on the jω axis on the s-plane, whereas ω_d is the digital frequency labeled on the unit circle in the z-plane.

We substitute $s = j\omega_a$ and $z = e^{j\omega_d T}$ into the BLT

$$j\omega_a = \frac{2}{T}\frac{e^{j\omega_d T} - 1}{e^{j\omega_d T} + 1}.$$

$$\omega_a = \frac{2}{T}\tan\left(\frac{\omega_d T}{2}\right).$$

$$\omega_d = \frac{2}{T}\tan^{-1}\left(\frac{\omega_a T}{2}\right).$$

Example:

The normalized lowpass filter with a cutoff frequency of 1 rad/sec is given as:

$$H_P(s) = \frac{1}{s+1}.$$

Use the given $H_p(s)$ and the BLT to design a corresponding digital IIR lowpass filter with a cutoff frequency of 15 Hz and a sampling rate of 90 Hz.

Solution:

First, we obtain the digital frequency as

$$\omega_d = 2\pi f = 2\pi(15) = 30\pi \text{ rad/sec, and } T = 1/f_s = 1/90 \text{ sec.}$$

We then follow the design procedure:

1. First calculate the prewarped analog frequency as

$$\omega_a = \frac{2}{T}\tan\left(\frac{\omega_d T}{2}\right) = \frac{2}{1/90}\tan\left(\frac{30\pi/90}{2}\right),$$

$$\text{that is, } \omega_a = 180 \times \tan(\pi/6) = 180 \times \tan(30^\circ) = 103.92 \text{ rad/sec.}$$

2. Then perform the prototype transformation (lowpass to lowpass) as follows:

$$H(s) = H_P(s)\big|_{s=\frac{s}{\omega_a}} = \frac{1}{\frac{s}{\omega_a}+1} = \frac{\omega_a}{s+\omega_a},$$

which yields an analog filter:

$$H(s) = \frac{103.92}{s + 103.92}.$$

3. Apply the BLT, which yields

$$H(z) = \left. \frac{103.92}{s + 103.92} \right|_{s = \frac{2}{T}\frac{z-1}{z+1}}.$$

We simplify the algebra by dividing both the numerator and the denominator by 180:

$$H(z) = \frac{103.92}{180 \times \frac{z-1}{z+1} + 103.92} = \frac{103.92/180}{\frac{z-1}{z+1} + 103.92/180} = \frac{0.5773}{\frac{z-1}{z+1} + 0.5773}.$$

Then we multiply both numerator and denominator by (z + 1) to obtain

$$H(z) = \frac{0.5773(z+1)}{\left(\frac{z-1}{z+1} + 0.5773\right)(z+1)} = \frac{0.5773z + 0.5773}{(z-1) + 0.5773(z+1)}$$
$$= \frac{0.5773z + 0.5773}{1.5773z - 0.4227}.$$

Finally, we divide both numerator and denominator by 1.5773z to get the transfer function in the standard format:

$$H(z) = \frac{(0.5773z + 0.5773)/(1.5773z)}{(1.5773z - 0.4227)/(1.5773z)} = \frac{0.3660 + 0.3660z^{-1}}{1 - 0.2679z^{-1}}.$$

7/References :-

1. Schaum's Outline of Theory and Problems of Digital Signal processing.
2. Digital signal processing, principles, algorithms, and applications by John G. Proakis and Dimitris G. Manolakis.
3. Signal and systems, Alan Oppenheim.