Classical methods

E- Frequency response

In sinusoidal circuit analysis, we have learned how to find voltages and currents in a circuit with a constant frequency source. If we left the amplitude of the sinusoidal source remain constant and vary the frequency, we obtain the circuit's ***frequency response.*** The frequency response may be regarded as a complete description of the sinusoidal steady state behavior of a circuit as a function of frequency.

The frequency response of a circuit is the variation in its behavior with change in signal frequency.

The Bode plots

Bode plots are semi log plots of the magnitude (in decibels) and phase (in degrees) of a transfer function versus frequency. The frequency range required in frequency response is often so wide that is inconvenient to use a linear scale for the frequency axis. Also, there is a more systematic way of locating the important features of the magnitude and the phase plots of transfer function. For these reasons, it has become standard practice to use a logarithmic scale for the frequency axis and a linear scale in each of the separate plots of magnitude and phase.

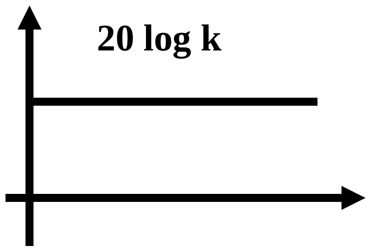
Note:

Identification of a linear system is based on

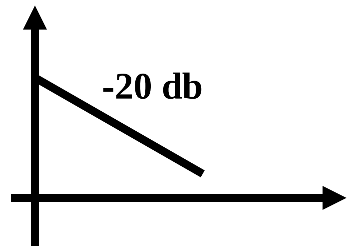
- Sine wave inputs are applied to the systems.
- Steady state output is observed (Magnitude ratio and phase).
- Use M and Φ plots to estimate the various break frequencies (poles and zeros) of the transfer function.

Note:

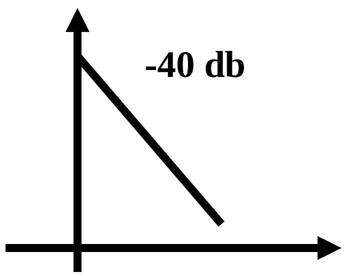
- Low frequency indicate the type of the system.
- Intermediate frequency indicates the existence of zeros.



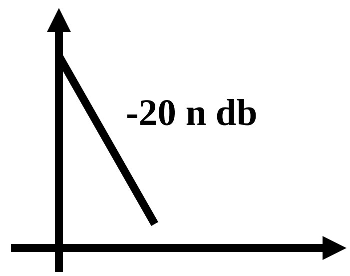
Type 0



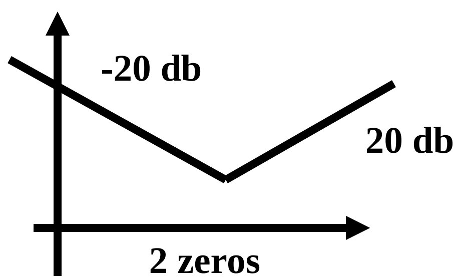
Type 1



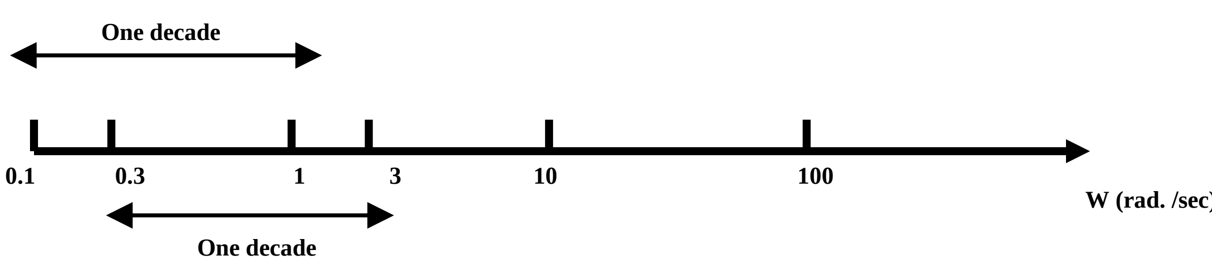
Type 2



Type n



- If the poles and zeros are too close, it is so difficult to estimate accurately their locations.
- A decade is the frequency band from ***w*** to ***10 w***.



- The following table provides a few gains with the corresponding values in decibels.

Magnitude	**20 $\log_{10}$ (db)**
0.001	**-60**
0.01	**-40**
0.1	**-20**
0.5	**-6**
$1/\sqrt{2}$	**-3**
1	**0**
$\sqrt{2}$	**3**
2	**6**
10	**20**
100	**40**
1000	**60**

- If ***gain > 1*** then ***+ve db.***
- If ***gain < 1*** then ***-ve db.***
- If ***gain = 1*** then ***0 db.***
- ***Φ = 0*** for +***ve gain.***
- ***Φ = -180*** for ***-ve gain.***

Example1: Estimate the transfer function of a system using Bode plot if the frequency response is given below:

w	**0.1**	**0.2**	**0.3**	**0.5**	**0.7**	**0.9**	**1**	**2**	**3**	**5**	**6**	**10**
M_{db}	**-9.5**	**-9.5**	**-10**	**-8**	**-5**	**-7**	**-8**	**-22**	**-30**	**-40**	**-45**	**-51**
φ	**-4.5**	**-9**	**-13.5**	**-31.5**	**-54**	**-99**	**-121.5**	**-162**	**-171**	**-171**	**-173.5**	**-175.5**

Solution:

*) The Dc gain can be calculated as:

20 log k = - 9.5

k = 0.334

From the phase plot dips asymptotically to -180 degree relative to the input. -180 degree is two multiples of -90 degrees, so the system is at least second order.

*) The natural frequency of a second order system, is the frequency when the phase is -90

$w_n = w_{-90}$, from bode plot $w_n = 0.88$ rad / sec

*) To find ζ (damping ratio)

$$G(jw) = \frac{k}{1 + 2 * \zeta \left(\frac{w}{w_n} j\right) + \left(j \frac{w}{w_n}\right)^2}$$

$$\left| G(jw) \right| = \frac{k}{\sqrt{\left(1 - .\frac{w^2}{w^2_n}\right)^2 + \left(2 \zeta \frac{w}{w_n}\right)^2}}$$

$$\left| G(jw) \right|_{-90} = \frac{.334}{\sqrt{(1-1)^2 + (2\zeta)^2}} = \frac{0.334}{2\zeta}$$

From magnitude plot $\left| G(jw) \right|_{-90} = -6.3$ db

$20 \log M = -6.3$ db

$M = 0.484$

$$\zeta = \frac{0.334}{2 * 0.484} = 0.345 \text{ (underdamped)}$$

Example2: Estimate the transfer function of a system using Bode plot if the frequency response is given below:

w	0.1	0.2	0.5	0.7	0.9	1	2	3	4	6	10	20	30
M_{db}	**19**	**12**	**04**	**-0.7**	**-5**	**-7.8**	**-20**	**-19.2**	**19**	**-22.2**	**-27.7**	**-32.5**	**-36**

Solution:

The Dc gain can be calculated as:

20 log k = - 1

k = 0.891

$$T.F = \frac{k\,(T_2\ s+1)^2}{s\,(T_1 s+1)\,(T_3 s+1)}$$

$$w_1 = 0.5\, r/s \Rightarrow T_1 = 2,$$

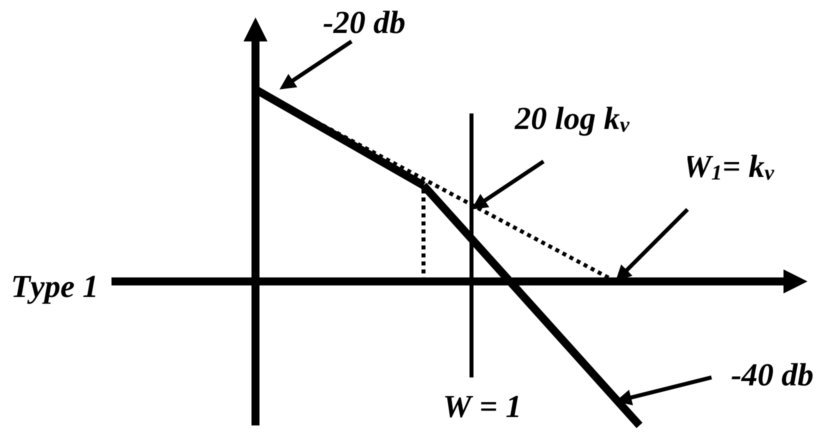
$$w_2 = 2.2\ r/s \Rightarrow T_2 = 0.45,$$

$$w_3 = 4.2\, r/s \Rightarrow T_3 = 0.24$$

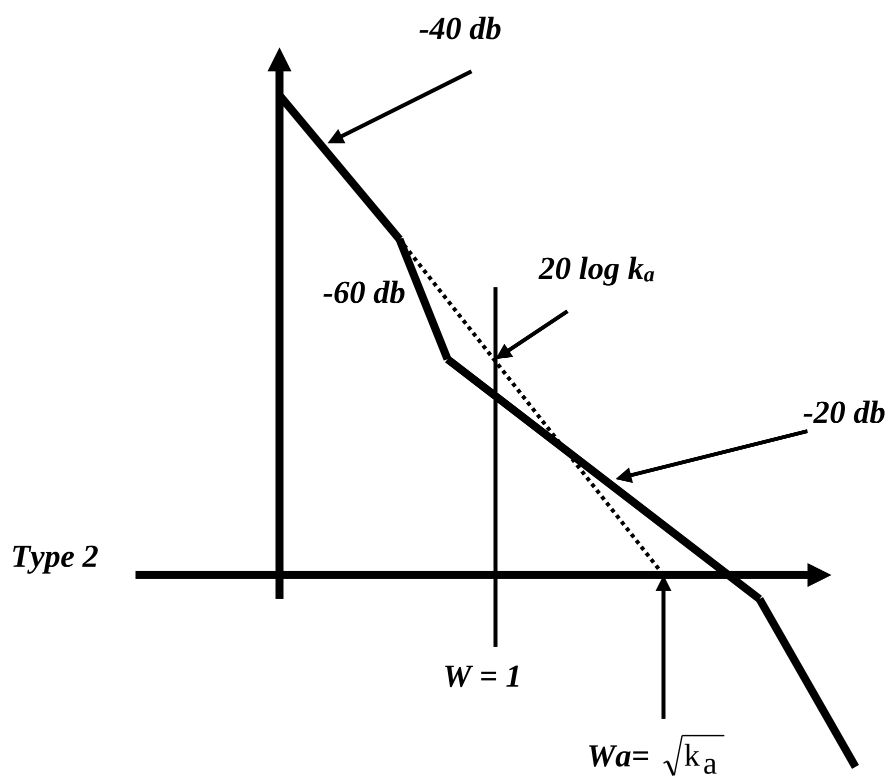
$$\therefore G(s) = \frac{0.891\,(0.45\,s+1)^2}{s\,(2\,s+1)\,(0.24\,s+1)}$$

Note:

*) Type1 system: the determination of the static velocity error coefficients is as shown in the figure below



*) Type2 system: the determination of the static acceleration error coefficients is as shown in the figure below



***<u>Homework</u>*:**

Q1/ The open loop frequency response of a linear system is given by the following measurements. Find the transfer function of the system.

W r/s	***0.2***	***0.8***	***1***	***1.5***	***2.5***	***3.5***	***6***	***10***	***15***	***26***	***30***	***40***	***60***	***80***
M db	***34***	***22***	***18***	***16***	***13***	***12.5***	***15***	***19***	***19.5***	***17***	***15***	***11***	***4***	***-2***

Q2/ given the data obtained from a frequency response of a system to a sinusoidal input, determine the transfer function of a plant G(S)

W r/s	**.1**	**.2**	**.6**	**1**	**1.5**	**2**	**2.5**	**4**	**5**	**7**	**9**	**20**	**35**	**60**	**100**
M db	**-19**	**-32**	**-46**	**-50**	**-55**	**-56**	**-58**	**-53**	**-51**	**-48**	**-46**	**-40**	**-46**	**-59**	**-68**

Q3/ The open loop frequency response of a linear system is given by the following measurements. Find the transfer function of the system.

w	**0.1**	**0.2**	**0.4**	**0.7**	**1**	**2**	**3**	**6**	**10**	**30**	**100**
M_{db}	**-12.3**	**-12.3**	**-15**	**-17.5**	**-20**	**-25**	**-30**	**-40**	**-45**	**-63**	**-85**
φ	**-13.5**	**-18.5**	**-45**	**-67.5**	**-81**	**-103.5**	**-112.5**	**-135**	**-153**	**-171**	**-175.5**

Q4/ Find the transfer function of the system from the following frequency response measurements.

W r/s	**.1**	**.2**	**.5**	**1**	**2**	**4**	**6**	**8**	**10**	**20**
M db	**35**	**30**	**20**	**14**	**6**	**-6**	**-12**	**-17**	**-21**	**-32**

Q5/ Find the transfer function of the system from the following frequency response measurements.

W r/s	.1	1	2	10	20	50	100	200
M db	60	20.9	11	-8	-15	-29	-41	-52.1

Q6/ Find the transfer function of the system from the following frequency response measurements.

w	0.1	0.5	1	3	5	10	30
M_{db}	50	36	30	17.5	10	-0.9	-19.5
φ	-92	-99.5	-108	-135	-149	-163	-174

<u>Off line parametric models</u>

In system identification both structure and the true parameters θ of a system may be prior unknown. Linear structures are very useful starting point in black box identification, and in most cases provide predications that are accurate enough. Since the structure is simple, it is also simple to validate the performance of the model. The selection of a model structure is largely based on experience and the information that is available of the process.

Similarly, parameters estimates $\hat{\theta}$ may be based on prior information concerning the process. If these parameters are not available, efficient techniques exist for estimating some or all of the unknown parameters using sampled data from the process. We shall concern with some methods related to the estimation of the parameters in a linear systems. These methods assumed that a set of input-output data pair is available, either Off line or Online, giving examples of the system behavior. If the order is not known “prior”, then we have to increase the number of the unknown parameters. There is always a trade off between the accuracy of estimating the order and the burden of computation. Consider the ***discrete time model*** from noise free I/p – O/p data with input u (k) and output y (k) described by Z transform

$$\frac{y(z)}{u(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ----+ b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + ------- + a_n z^{-n}} -------1$$

Where n ≥ m and z^{-q} back ward shift operator

- For most digital system $b_0 = 0$, i.e. the output is delayed from the input at least by one sample.
- Equation 1 can be written in the difference form as :

$$y(z)\ z^{-1} = y_{k-1}$$

$$\therefore y(k) + a_1\ y(k-1) + a_n\ y(k-n) = b_1\ u(k-1) + --- + b_m\ u(k-m)$$

$$\therefore y(k) = -\sum_{i=1}^{n} a_i\ y(k-i) + \sum_{i=1}^{m} b_i\ u(k-i) ------------2$$

It is required to estimate a_i^s and b_i^s from the input and output data

$$\begin{bmatrix} y_k \\ y_{k+1} \\ . \\ . \\ y_{k+p} \end{bmatrix} = \begin{bmatrix} -y_{k-1} ---- & -y_{k-n} & u_{k-1} & u_{k-m} \\ -y_k ---- & -y_{k-n+1} & u_k & u_{k-m+1} \\ & & & \\ -y_{k+p-1} ---- & -y_{k+p-n} & u_{k+p-1} & u_{k+p-m} \end{bmatrix} \begin{bmatrix} a1 \\ a2 \\ a3 \\ . \\ an \\ b1 \\ b2 \\ . \\ bm \end{bmatrix} -----3$$

OR $\underline{y} = \phi * \underline{\theta}$ ----------------------------4

Where

$\underline{y}$: *Is the output vector*

ϕ: *Is the data matrix*

θ : *is the unknown parameters vector*

P= n +m

$$\therefore \theta = \phi^{-1} * y \text{ ---------------------------------5}$$

And ϕ : *Is a square nonsingular matrix*

<u>Least Square method</u>

The method of least square is essential in system and control engineering. It provides a simple tool for estimating the parameters of the system by minimizing the model residuals, (ei) which is the difference between the measured output of the system and that of the model.

$$e_i = y_i \text{ (measured)} - y_i \text{ (model)} \text{ ---------------------------6}$$

The performance criterion is usually taken as the sum of squares of the residuals

$$v = \sum_{i=1}^{N} (e_i)^2 \text{ ---------------------------------------7}$$

And the estimation algorithm minimizes ***v*** by choosing $\underline{\hat{\theta}}$ such that

$$v = \min_{\hat{\theta}} (v) \text{ ---------------------------------------8}$$

The residual in equation (6) can be written as:

$$e_i = y_i \ (\text{measured}) - \phi * \underline{\hat{\theta}} \text{-----------------------------} 9$$

$$v = \sum e_i^2 = e^T e$$

$$v = [\, y - \phi * \underline{\hat{\theta}} \,]^T [\, y - \phi * \underline{\hat{\theta}} \,] \text{-----------------------------} 10$$

Where T is the transpose of a vector.

The minimization can be performed by differentiating with respect to θ and equating to zero.

$$\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta} (\, e^T e \,) = \frac{\partial e^T}{\partial \theta} e + \frac{\partial e^T}{\partial \theta} e$$

$$\therefore \frac{\partial v}{\partial \theta} = 2 \frac{\partial e^T}{\partial \theta} e$$

$$\frac{\partial v}{\partial \theta} = 2 * [\, \frac{\partial}{\partial \theta} (\, y - \phi \hat{\theta} \,)^T \,] [\, (\, y - \phi \hat{\theta} \,) \,]$$

$$\frac{\partial v}{\partial \theta} = -2 * \phi^T (\, y - \phi \hat{\theta} \,) = 0$$

$$\therefore -\phi^T y + \phi^T \phi \hat{\theta} = 0$$

$$\therefore \hat{\theta} = (\, \phi^T \phi \,)^{-1} \phi^T y \text{-----------------------------} 11$$

This is the basic least square algorithm and $\underline{\hat{\theta}}$ is the least square estimates (LSE).

Example: A system is described by the difference equation

$$y_i + a_1 y_{i-1} = b_1 u_{i-1}$$

Use the following data to find the estimates of the parameters a_1 and b_1 by the least square procedure

Time	1	2	3	4	5	6	7	8
y_i	0	1	1.8	2.44	0.95	1.76	0.41	-0.67
u_i	1	1	1	-1	1	-1	-1	1

Solution:

$$y_i = -a_1\, y_{i-1} + b_1\, u_{i-1} \qquad (i = 1, 2, 3, 4, 5, 6, 7, 8)$$

$$y = \phi\,\theta = \begin{bmatrix} -y_1 & u_1 \\ -y_2 & u_2 \\ -y_3 & u_3 \\ -y_4 & u_4 \\ -y_5 & u_5 \\ -y_6 & u_6 \\ -y_7 & u_7 \end{bmatrix} * \begin{bmatrix} a_1 \\ b_1 \end{bmatrix},$$

$$\phi^T\,\phi = \begin{bmatrix} 0 & -1 & -1.8 & -2.44 & -0.95 & -1.76 & -0.41 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \\ -1.8 & 1 \\ -2.44 & -1 \\ -0.95 & 1 \\ -1.76 & -1 \\ -0.41 & -1 \end{bmatrix}$$

$$\phi^T\,\phi = \begin{bmatrix} 14.362 & .86 \\ .86 & 7 \end{bmatrix} \Rightarrow (\,\phi^T\,\phi\,)^{-1} = \begin{bmatrix} 0.07 & -0.0086 \\ -0.0086 & 0.144 \end{bmatrix}$$

$$\phi^T y = \begin{bmatrix} 0 & -1 & -1.8 & -2.44 & -0.95 & -1.76 & -0.41 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1.8 \\ 2.44 \\ 0.95 \\ 1.76 \\ 0.41 \\ -0.67 \end{bmatrix} = \begin{bmatrix} -10.629 \\ 6.31 \end{bmatrix}$$

$$\hat{\theta} = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0.07 & -0.0086 \\ -0.0086 & 0.144 \end{bmatrix} \begin{bmatrix} -10.629 \\ 6.31 \end{bmatrix} = \begin{bmatrix} -0.8 \\ 1 \end{bmatrix}$$

Homework:

Q1/ Use the least square criterion to fit the linear model described by

$y_i + a_1 y_{i-1} = b_1 u_{i-1}$ $\quad i = 1, 2, 3, 4, 5, 6, 7$ to the data given below

Time	**1**	**2**	**3**	**4**	**5**	**6**	**7**
y_i	**33**	**42**	**45**	**51**	**53**	**61**	**62**
u_i	**4**	**4.5**	**5**	**5.5**	**6**	**6.5**	**7**

Ans: (a1 = 11.11, b1 = 24.656)

Q2/ Use the least square criterion to fit the linear model described by

$y_i + a_1 y_{i-1} = b_1 u_{i-1}$ $\quad i = 1, 2, 3, 4, 5, 6$ to the data given below

Time	**1**	**2**	**3**	**4**	**5**	**6**
y_i	**0.23**	**0.65**	**-0.4**	**0.05**	**-0.32**	**-0.75**
u_i	**-1**	**-1**	**1**	**1**	**-1**	**1**

Ans: (a1 = -0.235, b1 = 0.088))

Q3/ Write a program to estimate the unknown parameters of a system using least square method.