Statistical characteristics in frequency domain

Power density spectrum P.D.S

A ***time-domain*** graph shows how a signal changes over time, whereas a ***frequency-domain*** graph shows how much of the signal lies within each given frequency band over a range of frequencies.

A given function or signal can be converted between the time and frequency domains with a pair of mathematical operators called a ***transform***. An example is the ***Fourier transform (FT)***. Fourier transform of a function produces a frequency spectrum which contains all of the information about the original signal, but in a different form. The 'spectrum' of frequency components is the frequency domain representation of the signal. The ***inverse Fourier transform (IFT)*** converts the frequency domain function back to a time function.

The Fourier transformation pair

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(w) * e^{jwt}\, dw \text{ ----Inverse FT}$$

$$a(w) = \int_{-\infty}^{\infty} x(t) * e^{-jwt}\, dt \text{ ------FT}$$

The power density spectrum (P.D.S) of a signal measures the distribution of power in the signal over the frequency range of the signal.

Using the definition of acf and the fourier transformation pair, R_{XX} can be represented by the wiener-Khintchine relationship as following:

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{XX}(w) * e^{jw\tau} * dw ----\text{Inverse FT}$$

$$\phi_{XX}(w) = \int_{-\infty}^{\infty} R_{XX}(\tau) * e^{-jw\tau} * d\tau ------\text{FT}$$

Where ***$\Phi_{XX}(w)$*** is the P.D.S

Properties of $\Phi_{XX}(w)$

1- $R_{XX}(0) = E[X^2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_{XX}(w) * dw$

Prove:

When $\boldsymbol{\tau = 0}$

$$R_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{XX}(w) * e^{0} * dw$$

Also at : $w = 2\pi f$ that lead $dw = 2\pi\, df$

$$R_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{XX}(w) * 2\pi * df$$

$$\therefore R_{XX}(0) = \int_{-\infty}^{\infty} \phi_{XX}(2\pi f) * df = \int_{0}^{\infty} 2 * \phi_{XX}(2\pi f) * df$$

If

$$G_{XX}(f) = 2 * \phi_{XX}(2\pi f) \qquad \text{For } f \geq 0$$

$$\therefore R_{XX}(0) = \int_{0}^{\infty} G_{XX}(f)\, df$$

2- $\phi_{XX}(w)$ is a real function and $\phi_{XX}(w) \geq 0$

*) $$\phi_{XX}(w) = \int_{-\infty}^{\infty} R_{XX}(\tau) * e^{(-jw\tau)} * d\tau$$

$$\phi_{XX}(w) = \int_{-\infty}^{\infty} R_{XX}(\tau) * [\cos(w\tau) - j\sin(w\tau)] * d\tau$$

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$$\phi_{XX}(w) = \int_{-\infty}^{\infty} R_{XX}(\tau) * [\cos(w\tau)] * d\tau$$

$$\because G_{XX}(f) = 2 * \phi_{XX}(2\pi f)$$

$$\therefore \phi_{XX}(2\pi f) = \frac{G_{XX}(f)}{2}$$

$$\therefore \phi_{XX}(2\pi f) = 2\int_{0}^{\infty} R_{XX}(\tau) * [\cos(w\tau)] * d\tau$$

$$\therefore G_{XX}(f) = 4\int_{0}^{\infty} R_{XX}(\tau) * \cos(2\pi f \tau) * d\tau$$

*) $$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{XX}(w) * e^{jw\tau} * dw$$

$$R_{XX}(\tau) = \frac{1}{2\pi} * 2\int_{0}^{\infty} \frac{G_{XX}(f)}{2} * \cos(2\pi f \tau) * 2\pi * df$$

$$\therefore R_{XX}(\tau) = \int_{0}^{\infty} G_{XX}(f) * \cos(2\pi f \tau) * df$$

Final conclusion

$$\therefore R_{XX}(\tau) = \int_{0}^{\infty} G_{XX}(f) * \cos(2\pi f \tau) * df$$

$$\therefore G_{XX}(f) = 4\int_{0}^{\infty} R_{XX}(\tau) * \cos(2\pi f \tau) * d\tau$$

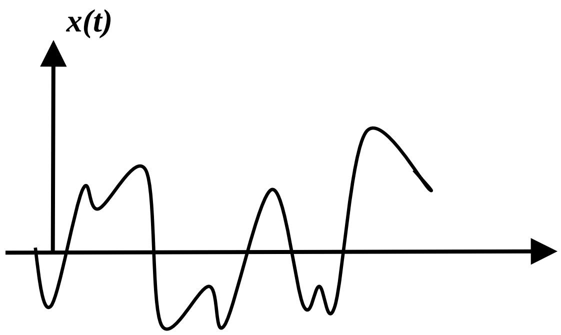
3- $\phi_{xx}(w) = \phi_{xx}(-w)$ it is even function

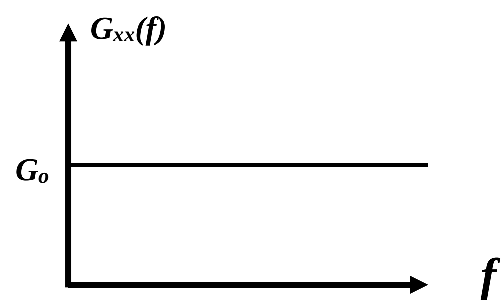
Note: The cross power spectrum $\phi_{xy}(w)$ of two continuous-time random processes x (t) and y (t) is defined as the Fourier transform of $R_{XY}(\tau)$

$$R_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{xy}(w) * e^{jw\tau} * dw ----\text{Inverse FT}$$

$$\phi_{xy}(w) = \int_{-\infty}^{\infty} R_{xy}(\tau) * e^{-jw\tau} * d\tau ------\text{FT}$$

Example1: The white noise has a constant power density G_o over an infinite band width, find Rxx(τ).





$$R_{xx}(\tau) = \int_{0}^{\infty} G_{xx}(f) * \cos(2\pi f \tau) * df$$

$$R_{xx}(\tau) = \int_{0}^{\infty} G_o * \cos(2\pi f \tau) * df$$

$$R_{xx}(\tau) = \frac{G_o}{2}\int_{-\infty}^{\infty} \cos(w\tau + j\sin(w\tau)) * df$$

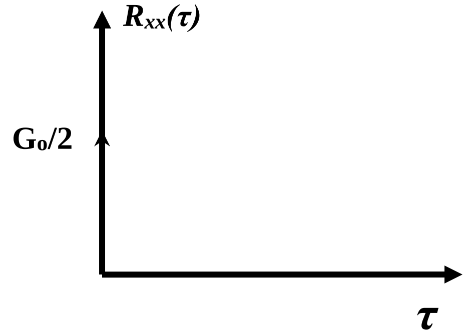
$$R_{xx}(\tau) = \frac{G_o}{2}\int_{-\infty}^{\infty} e^{jw\tau} * df$$

$$R_{xx}(\tau) = \frac{G_o}{4\pi}\int_{-\infty}^{\infty} e^{jw\tau} * dw$$

From the previous list

$$\because \delta(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jw\tau} * dw$$

$$\therefore R_{XX}(\tau) = \frac{G_o}{2} * \delta(\tau)$$



Example2: A signal has P.D.S $\Phi_{XX}(w) = \Phi_o / (1+w^2/ w^2_1)$,where $-\infty < w < +\infty$, find Rxx(0).

Note: $\int dx/(a^2 + b^2 x^2) = 1/(ab)*\tan^{-1} bx/a$

Solution:

$$R_{XX}(\tau) = \frac{2}{2\pi}\int_{0}^{\infty} \left(\frac{\phi_o}{1+\frac{w^2}{w_1^2}}\right) * e^{jw\tau} * dw$$

$$R_{XX}(0) = \frac{1}{\pi} * \int_{0}^{\infty} \left(\frac{\phi_o}{1+\frac{w^2}{w_1^2}}\right) * e^{0} * dw$$

$$R_{XX}(0) = \frac{w_1^2 * \phi_o}{\pi} * \int_{0}^{\infty} \left(\frac{1}{w_1^2 + w^2}\right) * dw$$

$$R_{XX}(0) = \frac{\phi_o * w_1^2}{\pi} * \frac{1}{w_1} * \tan^{-1}\frac{w}{w_1} \Big]_0^{\infty}$$

$$R_{XX}(0) = \frac{\phi_o * w_1}{\pi} * \left[\tan^{-1}\infty - \tan^{-1} 0\right] = \frac{\phi_o * w_1}{\pi} * \left[\frac{\pi}{2}\right]$$

$$R_{XX}(0) = \frac{\phi_o * w_1}{2}$$

Example3: Find Rxx(τ) of the sinusoidal signal whose P.D.S is

$$\phi_{xx}(w) = \frac{A^2}{2} * \delta(f - f_o).$$

Solution:

$$R_{xx}(\tau) = \frac{1}{2\pi} * \int_{-\infty}^{\infty} \phi_{xx}(w) * e^{jw\tau} * dw$$

$$R_{xx}(\tau) = \frac{1}{2\pi} * \int_{-\infty}^{\infty} \frac{A^2}{2} * \delta(f - f_o) * e^{j2\pi f\tau} * 2\pi * df$$

$$R_{xx}(\tau) = \frac{A^2}{2} * \int_{-\infty}^{\infty} \delta(f - f_o) * e^{j2\pi f\tau} * 2\pi * df$$

We note that $\delta(f - f_o) = 1$ when $f = f_o$ and $\delta(f - f_o) = 0$ else where

$$R_{xx}(\tau) = \frac{A^2}{2} * \int_{-\infty}^{\infty} \delta(f - f_o) * \cos(2\pi f\tau) * df$$

$$R_{xx}(\tau) = \frac{A^2}{2} * \cos(2\pi f_o \tau) = \frac{A^2}{2} * \cos(w_o \tau)$$

Example4: Find P.D.S for a sinusoidal signal whose autocorrelation function is:

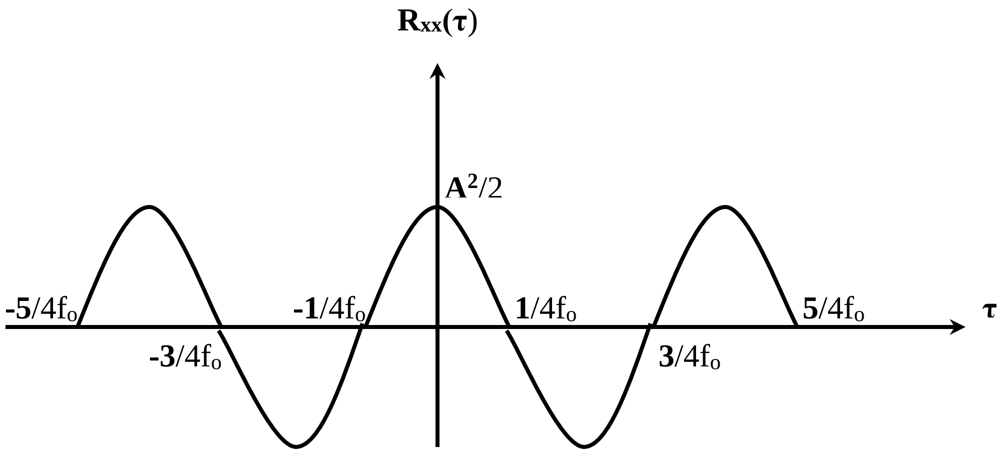
$R_{xx}(\tau) = \frac{A^2}{2} * \cos(w_o \tau)$. Draw ACF

Solution:

$\cos(\theta) = 0$ when $\theta = \frac{\pi}{2}(2n - 1)$ −−−−−−−− $n = 1,2,3..........$

$$\therefore w_o * \tau = \frac{\pi}{2} * (2n - 1)$$

$$2 * \pi * f_o * \tau = \frac{\pi}{2}(2n - 1) \Rightarrow \tau = \frac{(2n - 1)}{4 * f_0}$$



*) $\phi_{XX}(w) = \int_{-\infty}^{\infty} R_{XX(\tau)} * e^{-jw\tau} * d\tau$

$$\phi_{XX}(w) = \int_{-\infty}^{\infty} \frac{A^2}{2} * \cos(w_o * \tau) * e^{-jw\tau} * d\tau$$

$$\phi_{XX}(w) = \frac{A^2}{2} \int_{-\infty}^{\infty} (\cos(w_o * \tau) \pm j\sin(w_o * \tau)) * e^{-jw\tau} * d\tau$$

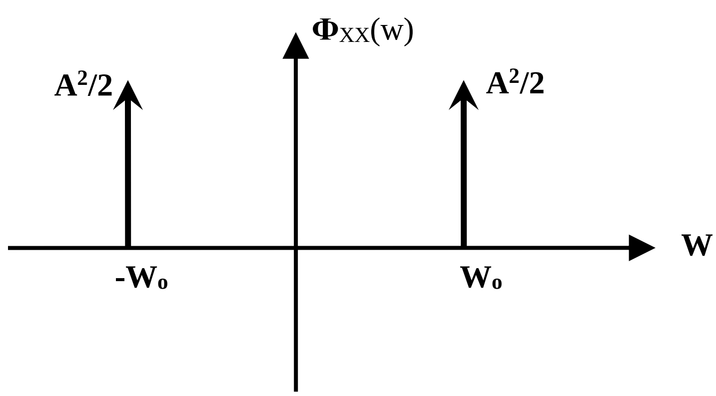
$$\phi_{XX}(w) = \frac{A^2}{2} \int_{-\infty}^{\infty} e^{\pm jw\tau} * e^{-jw\tau} * d\tau$$

$$\phi_{XX}(w) = \frac{A^2}{2} \int_{-\infty}^{\infty} e^{j(\pm w_o - w)\tau} * d\tau = \frac{A^2}{2} \int_{-\infty}^{\infty} e^{-j(w \mp w_o)\tau} * d\tau$$

From previous list

$$\delta(w) = \int_{-\infty}^{\infty} e^{-jw\tau} \, d\tau$$

$$\phi_{xx}(w) = \frac{A^2}{2} * \delta(w \mp w_o)$$



Example5: If the autocorrelation of a signal is equal to $R_{XX}(\tau) = V\cos(\tau\pi/T)$ prove that

$\Phi_{XX}(w) = VT/2\ [\ \text{sinc}(((\pi/T) - w)\,T/2) + \text{sinc}(((\pi/T) + w)\,T/2)\ ]$

Where $-T/2 \le \tau \le T/2$

Hint : * $\cos\Theta = 1/2\ [\ e^{j\Theta} + e^{-j\Theta}\ ]$

* $\sin\Theta = 1/j2\ [\ e^{j\Theta} - e^{-j\Theta}\ ]$

Solution:

$$\phi_{XX}(w) = \int_{-\infty}^{\infty} R_{XX(\tau)} * e^{-jw\tau} * d\tau$$

$$\phi_{XX}(w) = 2 * V \int_{0}^{T/2} \cos\left(\frac{\pi * \tau}{T}\right) * e^{-jw\tau} * d\tau$$

$$\phi_{XX}(w) = V \int_{0}^{T/2} \left(e^{j\left(\frac{\tau\pi}{T}\right)} + e^{-j\left(\frac{\tau\pi}{T}\right)} \right) * e^{-jw\tau} * d\tau$$

$$\phi_{XX}(w) = V \int_{0}^{T/2} \left(e^{j\left(\frac{\pi}{T} - w\right)\tau} \right) d\tau + V \int_{0}^{T/2} \left(e^{-j\left(\frac{\pi}{T} + w\right)\tau} \right) d\tau$$

Let

$$A = V \int_{0}^{T/2} \left(e^{j\left(\frac{\pi}{T} - w\right)\tau} \right) d\tau = V \left[\sin\left(\frac{\pi}{T} - w\right)\frac{T}{2} - \sin 0 \right] * \frac{1}{\frac{\pi}{T} - w}$$

$$A = \frac{VT}{2} * \frac{\sin\left(\frac{\pi}{T} - w\right)\frac{T}{2}}{\left(\frac{\pi}{T} - w\right)\frac{T}{2}} = \frac{VT}{2} * \ \text{sinc}\left(\frac{\pi}{T} - w\right)\frac{T}{2}$$

Let

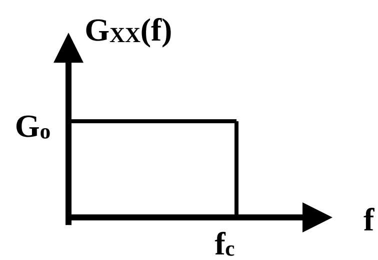
$$B = V \int_{0}^{T/2} \left(e^{-j\left(\frac{\pi}{T} + w\right)\tau} \right) d\tau = V \left[\sin\left(\frac{\pi}{T} + w\right)\frac{T}{2} - \sin 0 \right] * \frac{1}{\frac{\pi}{T} + w}$$

$$B = \frac{VT}{2} * \frac{\sin\left(\frac{\pi}{T}+w\right)\frac{T}{2}}{\left(\frac{\pi}{T}+w\right)\frac{T}{2}} = \frac{VT}{2} * \operatorname{sinc}\left(\frac{\pi}{T}+w\right)\frac{T}{2}$$

$$\therefore \phi_{XX}(w) = A + B = \frac{VT}{2} * \left[\operatorname{sinc}\left(\frac{\pi}{T}-w\right)\frac{T}{2} + \operatorname{sinc}\left(\frac{\pi}{T}+w\right)\frac{T}{2}\right]$$

<u>Homework</u>:

<u>Q1</u>: The low band pass filter has P.D.S as following, find and draw $R_{XX}(\tau)$



<u>Ans:</u> $R_{XX}(\tau) = \frac{G_O}{2\pi\tau} * \sin(2\pi\tau f_C)$

<u>Q</u>2: / A random binary sequence with $R_{XX}(\tau)$ equal to

$$R_{XX}(\tau) = \begin{cases} V^2 * \left[1 - \frac{|\tau|}{T}\right] & -T \le \tau \le T \\ 0 & \text{elsewhere} \end{cases}$$

Find and sketch $R_{XX}(\tau)$ and **$G_{XX}(f)$**

<u>Ans:</u> $\phi_{XX}(w) = V^2\, T \frac{\left[\sin\left(\frac{wT}{2}\right)\right]^2}{\left(\frac{wT}{2}\right)^2}$

<u>Q</u>3: / If the autocorrelation of a signal is shown in the figure below, prove that

$$\Phi_{XX}(W) = e^{-jw/2}\ \text{sinc}\ (w/2)$$

<u>Hint:</u> *$* \sin\Theta = 1/j2\ [e^{j\Theta} - e^{-j\Theta}]$*