Stochastic system

They are systems in which the time variables change randomly. The variables can be characterized by a probability function (i.e. they are statistically related)

- ***Mean or Expected value:***

The ***mean*** is the most common measure of central tendency and the one that can be mathematically manipulated. Simply, the mean is computed by summing all the scores in the distribution and dividing that sum by the total number of scores.

$$\mu = E(X) = \bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i \text{ -------------- discrete}$$

Where X_i, i= 1, 2, 3, --N are the measurements of stationary process and N; is the number of samples in the list.

The mean of a stationary random signal, X (t) is given by averaging over time:

$$\mu = E(X) = \bar{X} = \frac{1}{T-0}\int_{0}^{T} X(t)dt \text{ --------continuous}$$

Example

If you have a signal that is represented by:

V (t) = 2 + 12sin (2π500t)

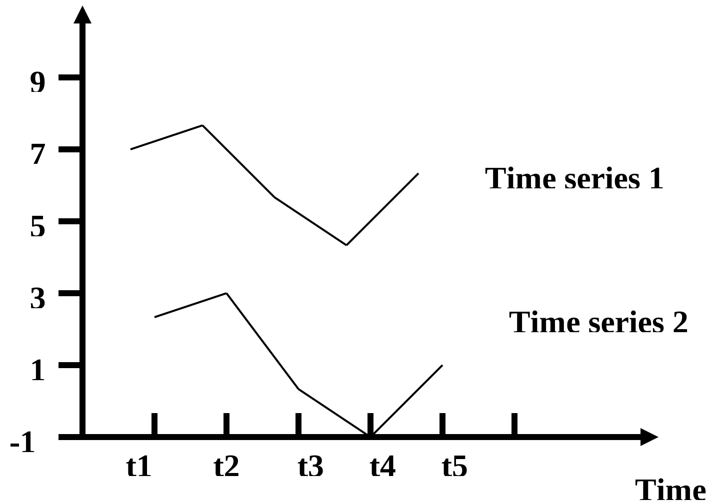
In this example, the simplest thing is to integrate from t = 0 to t= T, noticing that the signal repeats 500 times a second so the period is .002sec. Doing the integration,

$$\mu = \frac{1}{T}\int_{0}^{T} [2 + 12\sin(2*\pi*500*t)]dt$$

$$\mu = \frac{1}{T}\int_0^T [2*dt + 12*\frac{1}{T}\int_0^T \sin(2*\pi*500*t)]dt$$

$$\mu = 2$$

Example: Determine the Mean value for each time series



	t1	**t2**	**t3**	**t4**	**t5**
Time series1	***7***	***8***	***5***	***4***	***6***
Time sreies2	***2***	***3***	***0***	***-1***	***1***

For time series1: μ= (7 +8 +5+4 +6) / 5 = 6

For time series2: μ = (2 +3 + 0 + (-1) +1) /5 =1

- ***Variance***

To obtain the variance start by subtracting the average from each data item. Since there will be about as many items above average as below average, the resulting list of numbers will have about as many positive values as negative values. (In fact this list of deviations-from-average must itself average to zero!) Square each deviation, and proceed to find the average of the squared-deviations. The result is the ***variance*** which is sometimes called the ***mean square deviation,***

$$\sigma^2 = \mathrm{var}(x) = \frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})^2$$

Or

$$\sigma^2 = \mathrm{var}(x) = \frac{1}{T}\int_{0}^{T}(x(t) - \bar{X})^2 dt$$

The most common way to describe the range of variation is **standard deviation** (usually denoted by the Greek letter sigma: σ or ***SD***). The standard deviation is simply the ***positive square root*** of the **variance**, so we first start by describing the variance; take its square root to get the standard deviation.

Homework

Prove that

- ***For discrete signal***

$$\sigma^2 = \left(\frac{1}{N}\sum_{i=1}^{N}X_i^2\right) - \bar{X}^2$$

- ***For continuous signal***

$$\sigma^2 = [\frac{1}{T}\int_0^T X^2(t)dt] - \bar{X}^2$$

Example:

Calculate the SD and σ^2 for the following data 7, 8, 11, 6, 13, 8, 10

$\bar{X}$ = (7+8+11+6+13+8+10) / 7=9

$$\sigma^2 = \frac{1}{7}\sum_{i=1}^{7}(X_i - \bar{X})^2$$

$$\sigma^2 = \frac{1}{7}[(6-9)^2 + (7-9)^2 + (8-9)^2 + (8-9)^2 + (10-9)^2 + (11-9)^2 + 13-9)^2] = 5.14$$

$$SD = \sqrt{5.14} = 2.27$$

Homework

Q1/ Solve the same problem using

$\sigma^2 = E(X^{2)} - [E(X)]^2$

Q2/ The first of two groups has 100 items with mean 45 and variance 49. If the combined group has 250 items with mean 51 and variance 130. Find the mean and standard deviation of the second group.

Ans.: (12)

Q3/ The mean and standard deviation of 50 item were found to be 75 and 10 respectively. At the time of checking it was found that one item 65 was incorrect. Calculate the mean and SD if the wrong item is omitted.

Ans.: (75.204, 9.99)

- ***Auto correlation function (acf)***

It is a mathematical tool for finding repeating patterns, such as the presence of a periodic signal which has been buried under noise, or identifying the missing fundamental frequency in a signal implied by its harmonic frequencies. It is often used in signal processing for analyzing functions.

$$R_{XX}(t_1, t_2) = E[X(t_1) * X(t_2)]$$

$$\tau = |t_2 - t_1|$$

$$R_{XX}(\tau) = \frac{1}{T-0}\int_0^T X(t) * X(t+\tau)dt \text{ ------- continuous}$$

$$R_{XX}(K) = \frac{1}{N}\sum_{i=1}^{N} X_i * X_{i+k} \text{ ------------discrete}$$

$$K = 0,1,2,3,----$$

Properties of acf

****)*** $|R_{XX}(k,\tau)| \le R_{XX}(0)$ for all k or τ

****)*** $R_{XX}(k,\tau) = R_{XX}(-k,-\tau)$,

That is because acf is *even* function and symmetric around Y-axis

****)*** $R_{XX}(0) = E[X^2(t)] = \overline{X^2}$

Example:

Find the acf RXX (-1), R_{XX} (-2), R_{XX} (1), R_{XX} (2), from the following value. Then find R_{XX} (0)

t	1	2	3	4	5
X(t)	1.6	-3.7	7.2	2	-1

R_{XX} (-1) =1/4 (-3.7*1.6+7.2*-3.7+2*7.2+2*-1) = -5.04

R_{XX} (1) =1/4 (-3.7*1.6+7.2*-3.7+2*7.2+2*-1) = -5.04

R_{XX} (-2) =1/3 (7.2*1.6+2*-3.7+7.2*-1) = -1.03

R_{XX} (2) =1/3 (1.6*7.2+2*-3.7+7.2*-1) = -1.03

R_{XX} (0) =1/5 (1.6*1.6+3.7*3.7+7.2*7.2+2*2+1) = 14.62

Note:

****)*** In some cases, only a limited length of data is available, so that there is loss at the end as the time shift **τ** is increased from zero, so:

$$\hat{R}_{XX}(\tau) = \frac{1}{T-\tau}\int_{0}^{T-\tau} X(t) * X(t+\tau)dt -------continuous$$

Or digitally for limited length of data

$$\hat{R}_{XX} = \frac{1}{N-K}\sum_{i=1}^{N-1} X_i * X_{i+k} ------------discrete$$

****)*** When the acf is large at lag=0 (i.e. **τ** =0) and approximately zero elsewhere, this mean that it is generated from a signal which is normally distributed with mean=0 and variance= σ^2. A series of these properties is called the white noise otherwise it is a color noise.

- ***Cross correlation function (ccf)***

Measures the strength of relationship between two stationary signals while they are (k) sampling interval or **τ** apart.

$$R_{XY}(t_1, t_2) = E[X(t_1) * Y(t_2)]$$
$$\tau = |t_2 - t_1|$$
$$R_{XY}(\tau) = \frac{1}{T-0}\int_0^T X(t) * Y(t+\tau)dt ------- \text{continuous}$$
$$R_{XY}(K) = \frac{1}{N}\sum_{i=1}^{N} X_i * Y_{i+k} ------------ \text{discrete}$$
$$K = 0,1,2,3,----$$

For limited length of data

$$\hat{R}_{XY}(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} X(t) * Y(t+\tau)dt ------- \text{continuous}$$

And

$$\hat{R}_{XY}(K) = \frac{1}{N-K}\sum_{i=1}^{N-K} X_i * Y_{i+k} ------------ \text{discrete}$$

Properties of ccf

***)** $R_{XY}(0)$ is not necessary > $R_{XY}(k, \tau)$

***)** $R_{XY}(\tau) \neq R_{XY}(-\tau)$

***)** $R_{XY}(\tau) = R_{YX}(-\tau)$

***)** $R_{XY}(\tau) \neq R_{YX}(\tau)$

Example:

Estimate $R_{XY}(-1)$, $R_{YX}(-2)$ and $R_{YX}(3)$ from the following data

t	1	2	3	4	5
X(t)	-1	1.5	-2	-3.5	-1
Y(t)	0.81	-2.28	-0.83	-0.65	-1.85

$R_{XY}(-1) = 1/4\ (1.5*.81+2.28*2+3.5*.83+.65) = 2$

$R_{YX}(-2) = 1/3\ (.83*1+1.5*-.65+2*1.85) = 1.835$

$R_{YX}(3) = ½\ (.81*-3.5 + 2.28*1) = -0.277$

Homework

Q1/ Find RXY (- 2), $R_{XY}(3)$, $R_{YX}(2)$, $R_{YX}(-1)$

t	1	2	3	4	5
X(t)	0.5	3	-2.4	1	-1.75
Y(t)	-0.25	3.6	1.6	0.95	2

Q2/ Find $R_{XY}(-3)$, $R_{YY}(3)$, $R_{YX}(2)$, $R_{XY}(3)$ from the following table

t	1	2	3	4	5	6	7	8
X(t)	0.1	0.3	0.4	1.6	3	3.4	4	2
Y(t)	1	1	2	3	2	1	3	4

Q3/ Estimate cross correlation functions $R_{XY}(1)$, $R_{YX}(1)$, $R_{YY}(2)$, $R_{XX}(-2)$

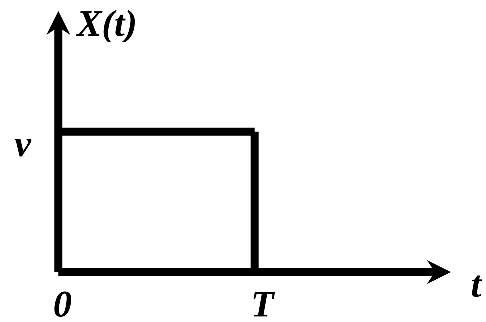
t	1	2	3	4	5
X(t)	0.4	- 3.2	- 2.2	1.4	3.4
Y(t)	- 0.8	- 2.2	- 1.8	- 0.8	-1.8

Q4 / A random binary sequence with

$$R_{XX}(\tau) = V^2[1-\frac{|\tau|}{T}]------------\text{for}(-T \le \tau \le T)$$

Sketch RXX (τ)

Q5 / Find the auto correlation function of the signal shown below



Q6 / Find the auto correlation function of the signal shown below

X (t) = V sin (w t +θ) *Ans.:* $V^2/2 * \cos(w\tau)$

- ***Some important rules***

1. $e^{jx} = \cos x + j \sin x$
2. $\cos x = 1/2 *(e^{jx} + e^{-jx})$
3. $\sin x = 1/2j *(e^{jx} - e^{-jx})$
4. $\cos^2 x + \sin^2 x =1$
5. $\cos^2 x = 1/2 (1 + \cos 2x)$
6. $\sin^2 x = 1/2 (1 - \cos 2x)$
7. $\cos x * \cos y = 1/2 [\cos (x-y) + \cos (x +y)]$
8. $\sin x * \sin y = 1/2 [\cos (x-y) - \cos (x +y)]$
9. $\sin x * \cos y = 1/2 [\sin (x+y) + \sin (x -y)]$
10. $\delta(w) = \int_{-\infty}^{\infty} e^{-jw} d\tau$
11. $\delta(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{jw\tau} dw$
12. $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$